Introduction to Ocean Waves

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Preface

Wind waves, with periods of a few seconds, and the tides, with periods of twelve hours or more, are really two examples of the same physical phenomenon. They differ only in the source of their energy. For the shortest-period waves—periods of, say, one to four seconds—the connection between the wind and the waves is obvious. On a windless day, the surface of Mission Bay is dead flat. But if the wind begins to blow, waves appear within a few minutes and grow steadily in amplitude to a saturation level that depends on the strength of the wind. When the wind stops blowing, the waves gradually decay.

For the breakers on Scripps Beach, the connection with the wind is not so obvious, but these longer-period waves—periods of, typically, ten seconds—are wind-generated too. Their wind sources are powerful storms that may have occurred days ago and thousands of miles away. Only the very strong winds associated with these storms can generate these long, fast-moving waves. Because their wavelengths are so long, these waves experience very little dissipation; they lose little of their energy on their long cross-ocean trip to San Diego. The energy in these long waves travels at a speed that increases with the wavelength. Because of this, these far-traveling waves sort themselves by wavelength during the long trip, with the longest waves reaching San Diego first. This sorting explains why the breakers on Scripps Beach often seem to have a single, well-defined period.

In contrast to the wind waves, the tides receive their energy from the gravitational pull of the Sun and Moon. This energy source imposes a scale—a wavelength—that is comparable to the Earth's radius and therefore not directly observable by eye. However, ancient peoples recognized the connection between astronomy and the tides merely by observing the regular periods of the tides. Because tidal periods are comparable to, or longer than, a day, the tides are strongly affected by the Earth's rotation. Because the spatial scale of the tides is so large, the tidal response to astronomical forcing is very sensitively dependent on the irregular shape of the ocean basins. Until very recently, this fact, and a misunderstanding of tidal dissipation mechanisms, defeated attempts at a quantitative physical explanation of the tides.

This course was originally planned to cover both wind waves and tides. However, it was soon realized that ten weeks is barely to sufficient to cover either of these topics in any detail. Wind waves were selected as being of greater general interest.

You can find lots of books about ocean waves. Nearly all of them fall into one of two categories: popular books full of pictures and sea lore, and textbooks written for people with a bachelor's degree in a physical science. All of the latter assume a prior knowledge of fluid mechanics, or contain a general introduction to fluid mechanics as part of the book.

In a ten-week course, we cannot afford to learn fluid mechanics before embarking on waves. Therefore, this textbook, which has been written especially for the course, avoids the need for a background in fluid mechanics by basing our study of waves on two fundamental postulates. These two postulates are:

1. The dispersion relation for ocean waves, which is introduced and explained in chapter 1.

2. The principle of wave superposition, which is explained and illustrated in chapters 2 and 3.

Strictly speaking, these two postulates apply only to ocean waves of very small amplitude. Nevertheless, a great many useful facts may be deduced from them, as we shall see in chapters 4, 5, 6 and 7. Not until chapter 8 do we justify our two postulates on the basis of first physical principles—the conservation of mass and momentum by the fluid. Chapter 8 is a whirlwind introduction to fluid mechanics, but its primary goal is a very limited one: to justify the two fundamental postulates that will have already proved so useful. Chapters 9 and 10 apply the newly derived equations of fluid mechanics to tsunamis and to the physics of the surf zone. However, no great expertise in fluid mechanics is required for this course. In fact, the course is designed to give you just a taste of that subject, enough to decide if you want to learn more about it.

However, you do need to know some math: basic differential and integral calculus, a bit of vector calculus, a good bit about ordinary differential equations, and at least a wee bit about partial differential equations. To determine if your math background is sufficient, have a look at the first few chapters.

Contents

Chapter 1

Basic waves

To describe ocean waves, we use a right-handed, Cartesian coordinate system in which the z-axis points upward. The x- and y-axes point in horizontal directions at right angles. In the state of rest, the ocean surface coincides with $z = 0$. When waves are present, the surface is located at $z = \eta(x, y, t)$, where t is time. The ocean bottom is flat, and it is located at $z = -H$, where H is a constant equal to the depth of the ocean. Refer to figure 1.1.

Our first basic postulate is this:

Postulate #1. If $A|k| \ll 1$, then the equation

$$\eta = A\cos(kx - \omega t) \tag{1.1}$$

describes a single, basic wave moving in the x-direction, where A, k, and ω are constants; and ω and k are related by

$$\omega = \sqrt{gk\tanh(kH)}. \tag{1.2}$$

Here, $g = 9.8$ m sec^{-2} is the gravity constant, and

$$\tanh(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}} \tag{1.3}$$

is the hyperbolic tangent function. This first postulate, summarized by (1.1) and (1.2), needs some elaboration and a lot of explaining. It will take us a while to do this. But before saying anything more about Postulate #1, we go on to state:

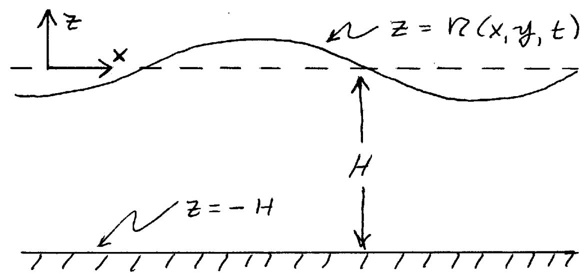

Figure 1.1: Ocean surface elevation in a basic wave.

Postulate #2. Still assuming $A|k| \ll 1$, we may add together as many waves satisfying Postulate #1 as we like; the result will be a physically valid motion. For example,

$$\eta = A_1 \cos(k_1 x - \omega_1 t) + A_2 \cos(k_2 x - \omega_2 t) \tag{1.4}$$

is physically valid if the pairs (k_1, ω_1) and (k_2, ω_2) each satisfy the requirement (1.2), that is, if

$$\omega_1 = \sqrt{g k_1 \tanh(k_1 H)} \quad \text{and} \quad \omega_2 = \sqrt{g k_2 \tanh(k_2 H)} \tag{1.5}$$

Using only these two postulates, we can explain quite a lot about ocean waves. Eventually we shall go deeper into the underlying physics, but for quite some time, these two postulates will suffice. Our immediate task is an elaboration of Postulate #1.

Equation (1.1) describes a single basic wave with amplitude A, wavenumber k, and frequency ω. The frequency ω is always positive; the wavenumber k can be positive or negative. If k is positive, then the wave moves to the right, toward positive x. If k is negative, the wave moves to the left. The *wave height*—the vertical distance between the crest and the trough—is equal to $2A$. See figure 1.2.

The wavelength λ is related to the wavenumber k by

$$\lambda = \frac{2\pi}{|k|} \tag{1.6}$$

and the wave period T is related to the frequency ω by

$$T = \frac{2\pi}{\omega} \tag{1.7}$$

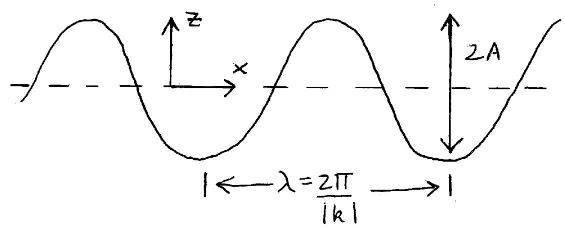

Figure 1.2: The wave height is twice the wave amplitude A.

The wave described by (1.1) moves in the x-direction at the phase speed

$$c = \frac{\omega}{k} \tag{1.8}$$

which has the same sign as k.

The restriction $A|k| \ll 1$ is an important one. It says that the wave height must be small compared to the wavelength. In other words, the wave must have a small slope. Only then are (1.1) and (1.2) an accurate description of a physical wave. The restriction to small amplitude A means that we are considering what oceanographers call *linear waves.* (The logic behind this terminology will be explained later on.) The theory of linear waves cannot explain such things as wave breaking or the transfer of energy between one wave and another. Nevertheless, and because nonlinear wave theory is so much more difficult, this course is largely limited to linear waves.

There is a further restriction on (1.1) and (1.2) which must be explained. These equations assume that the wave is neither being forced nor dissipated. That is, (1.1) and especially (1.2) describe a *free* wave. The equations apply best to the long ocean swells between the point at which they are generated by storms and the point at which they dissipate by breaking on a beach.

Equation (1.1) could be considered a general description of almost any type of wave, depending only on the interpretation of η. It is the *dispersion relation* (1.2) that asserts the physics, and tells us that we are considering a water wave. The dispersion relation is a relation between the frequency ω and the wavenumber k. Alternatively it can be considered a relation between the phase speed c and the wavelength λ.

The physical description (1.1-2) is incomplete. To have a complete description, we must specify how the fluid velocity depends on location and time. The fluid velocity is a vector field that depends on (x, y, z, t). We write it as

$$\mathbf{v}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)) \tag{1.9}$$

For the wave described by (1.1) and (1.2), the y-component of $\mathbf{v}$ vanishes. That is, $v = 0$; there is no velocity out of the page. The x- and z-components are given by

$$u = A\omega \frac{\cosh(k(H+z))}{\sinh(kH)} \cos(kx - \omega t) \tag{1.10}$$

and

$$w = A\omega \frac{\sinh(k(H+z))}{\sinh(kH)} \sin(kx - \omega t) \tag{1.11}$$

These are somewhat complicated expressions. However, two limiting cases will claim most of our attention.

From now on, we let the wavenumber k be positive. This means that we are focusing on waves moving to the right, but the generalization to left-going waves is obvious. Our first limiting case is the case of *deep water waves*, in which $kH \gg 1$; the water depth is much greater than a wavelength. In the limit $kH \gg 1$,

$$\tanh(kH) = \frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}} \to \frac{e^{kH}}{e^{kH}} = 1 \tag{1.12}$$

$$\frac{\cosh(k(H+z))}{\sinh(kH)} = \frac{e^{k(H+z)} + e^{-k(H+z)}}{e^{kH} - e^{-kH}} \to \frac{e^{k(H+z)}}{e^{kH}} = e^{kz} \tag{1.13}$$

$$\frac{\sinh(k(H+z))}{\sinh(kH)} = \frac{e^{k(H+z)} - e^{-k(H+z)}}{e^{kH} - e^{-kH}} \to \frac{e^{k(H+z)}}{e^{kH}} = e^{kz} \tag{1.14}$$

Thus the deep-water wave is described by

$$\text{DW} \qquad \eta = A\cos(kx - \omega t) \tag{1.15a}$$

$$\text{DW} \qquad \omega = \sqrt{gk} \tag{1.15b}$$

$$\text{DW} \qquad u = A\omega e^{kz} \cos(kx - \omega t) \tag{1.15c}$$

$$\text{DW} \qquad w = A\omega e^{kz} \sin(kx - \omega t) \tag{1.15d}$$

The abbreviation DW is a reminder that the equation applies only to deep water waves.

Our second limiting case represents the opposite extreme. It is the case $kH \ll 1$ of shallow water waves, in which the depth H is much less than the wavelength. In the limit $kH \ll 1$,

$$
\begin{aligned}
\tanh(kH) &= \frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}} = \frac{(1 + kH + \cdots) - (1 - kH + \cdots)}{(1 + kH + \cdots) + (1 - kH + \cdots)} \\
&\to \frac{2kH}{2} = kH
\end{aligned} \tag{1.16}
$$

$$
\begin{aligned}
\frac{\cosh(k(H+z))}{\sinh(kH)} &= \frac{e^{k(H+z)} + e^{-k(H+z)}}{e^{kH} - e^{-kH}} \\
&= \frac{(1 + k(H+z) + \cdots) + (1 - k(H+z) + \cdots)}{(1 + kH + \cdots) - (1 - kH + \cdots)} \\
&\to \frac{2}{2kH} = \frac{1}{kH}
\end{aligned} \tag{1.17}
$$

$$
\begin{aligned}
\frac{\sinh(k(H+z))}{\sinh(kH)} &= \frac{e^{k(H+z)} - e^{-k(H+z)}}{e^{kH} - e^{-kH}} \\
&= \frac{(1 + k(H+z) + \cdots) - (1 - k(H+z) + \cdots)}{(1 + kH + \cdots) - (1 - kH + \cdots)} \\
&\to \frac{2k(H+z)}{2kH} = (1 + z/H)
\end{aligned} \tag{1.18}
$$

Thus the shallow-water wave is described by

$$\text{SW} \qquad \eta = A\cos(kx - \omega t) \tag{1.19a}$$

$$\text{SW} \qquad \omega = \sqrt{gH}k \tag{1.19b}$$

$$\text{SW} \qquad u = \frac{A\omega}{kH}\cos(kx - \omega t) \tag{1.19c}$$

$$\text{SW} \qquad w = A\omega(1 + z/H)\sin(kx - \omega t) \tag{1.19d}$$

Swell far out to sea certainly qualifies as DW. According to (1.15b), the DW phase speed—the speed of the wave crests and troughs—is given by

$$\text{DW} \qquad c = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \tag{1.20}$$

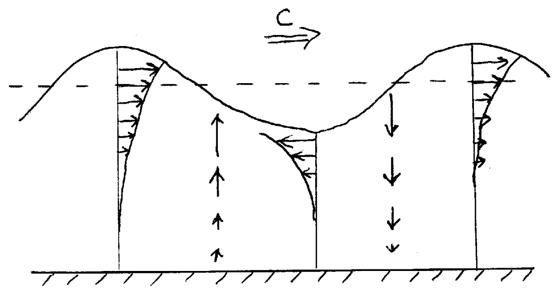

Figure 1.3: Fluid velocity in a deep-water wave moving to the right.

Hence, waves with longer wavelengths travel faster. The phase speed (1.20) is independent of the wave amplitude A, and it is much larger than the fluid velocity (1.15c-d). The latter is proportional to A and is therefore infinitesimally small in linear theory.

Because of the factor e^{kz} in (1.15c-d), the fluid velocity decays with distance below the surface (becoming smaller as z becomes more negative). According to (1.15c), the fluid velocity is in the direction of wave propagation under the crest, and in the opposite direction under the wave trough. According to (1.15d), the fluid is rising ahead of the crest, and descending behind it. See figure 1.3.

In the shallow-water limit (1.19), the horizontal velocity u is independent of z (figure 1.4). The vertical velocity w varies linearly with z, but it is smaller than u by a factor kH. The shallow-water phase speed $c = \sqrt{gH}$ depends on H but not on the wavelength λ. In shallow water, waves of all wavelengths move at the same speed.

Stand at the end of Scripps pier (if you can get past the gate) and watch the swells roll toward the beach. Because their vertical decay-scale is comparable to their wavelength, the waves extend downward a distance comparable to the spacing between wave crests. Where the ocean depth is greater than

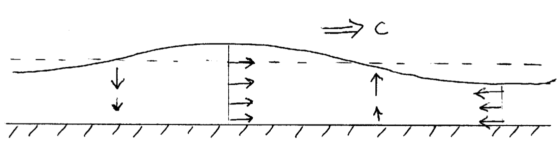

Figure 1.4: Fluid velocity in a shallow-water wave moving to the right.

a wavelength, the depth might as well be infinite; the waves don't feel the bottom. But where the ocean depth becomes smaller than a wavelength, the DW description becomes inaccurate, and we must pass over to the general description (1.1-2,1.10,1.11), which is valid for arbitrary H. When the waves reach shallow water—H much less than a wavelength—the simpler, SW description (1.19) applies.

The foregoing paragraph ignores a subtlety in all the preceding equations: Strictly speaking, these equations apply only to the situation in which H is a constant. We may talk about deep water, and we may talk about shallow water, but we cannot—strictly speaking—use these equations to describe a situation in which H varies. However, common sense suggests that we may use our general description in the case where H changes very gradually. If the mean water depth H changes by only a small percentage in a wavelength, then the wave ought to behave as if the depth were constant at its local value. This turns out to be correct.

Let x be the perpendicular distance toward shore. Let the mean water depth $H(x)$ decrease gradually in the x-direction. Then x-directed, incoming waves ought to obey the *slowly varying* dispersion relation,

$$\omega = \sqrt{g\, k(x) \tanh(k(x)H(x))} \tag{1.21}$$

obtained by replacing k with $k(x)$, and H with $H(x)$, in (1.2). As $H(x)$ decreases toward the shore, $k(x)$ must increase; the wavelength shortens as the wave shoals. (To see this, take the derivative of (1.21) to show that dk/dx and dH/dx must have opposite signs.) In very shallow water (1.21) becomes

$$\text{SW} \qquad \omega = \sqrt{gH(x)}k \tag{1.22}$$

and the phase speed $c = \sqrt{gH(x)}$.

Why, you may ask, is it the wavenumber k and not the frequency ω that must change to compensate the change in H? The frequency is proportional to the rate at which wave crests pass a fixed point. Consider any two fixed points—any two values of x. If the frequency were different at the two points, then the number of wave crests between them would continually increase or decrease (depending on which point had the higher frequency). No steady state could exist. This is not the situation at the beach. Hence ω must be constant. (In chapter 6 we will prove that this is so, provided that the waves have small enough amplitudes, and that the depth changes gradually on the scale of a wavelength.)

Back on the pier, look closely to see if this is true. As the waves approach shallow water, the wavelengths really do get shorter. The phase speed really does decrease. Later, when we develop this slowly varying theory more completely, we will show that other things are happening too. In particular, the wave amplitude A increases as the wave energy is squeezed into a smaller depth.

Now lean over the pier and drop an object into the water. Observe where it goes. If the object is a surfboard, it may move at the phase speed all the way to the beach. If the object is a piece of tissue paper, it will move forward with each wave crest, and backward with each wave trough, with a very slow net movement toward the shore. The tissue paper is moving with the water—not with the wave. Its velocity is the same as the velocity of the surrounding fluid particles. What, then, is the trajectory of the fluid particles?

First of all, what do we mean by a *fluid particle*? Fluid mechanicists usually ignore the fact that a fluid is composed of molecules and instead regard the fluid as a continuum—a continuous distribution of mass and velocity in space. This, it turns out, is a valid idealization if the fluid's molecules collide with each other frequently enough. It is the continuum velocity to which (1.15c-d) refer. By fluid particle, we mean an arbitrarily small piece of this continuum.

Let $(x_p(t), z_p(t))$ be the coordinates of a particular fluid particle, selected arbitrarily. We find the motion of the fluid particle by solving the coupled ordinary differential equations

$$\frac{dx_p}{dt} = u(x_p(t), z_p(t), t) \tag{1.23a}$$

$$\frac{dz_p}{dt} = w(x_p(t), z_p(t), t) \tag{1.23b}$$

For DW, the velocity fields are given by (1.15c-d), so we must solve

$$\text{DW} \qquad \frac{dx_p}{dt} = A\omega e^{kz_p} \cos(kx_p - \omega t) \tag{1.24a}$$

$$\text{DW} \qquad \frac{dz_p}{dt} = A\omega e^{kz_p} \sin(kx_p - \omega t) \tag{1.24b}$$

The *exact* solution of (1.24) is quite difficult (and also somewhat pointless because the right-hand sides of (1.24) are themselves approximations, valid only for small A). However, if A is small, the fluid particle never gets far from where it started. We can use this fact to justify an approximation that makes it easy to solve (1.24).

Let (x_0, z_0) be the *average* location of the fluid particle. Then

$$x_p(t) = x_0 + \delta x(t) \tag{1.25a}$$

$$z_p(t) = z_0 + \delta z(t) \tag{1.25b}$$

where $(\delta x(t), \delta z(t))$ is the small displacement of the fluid particle from its average location. Substituting (1.25) into (1.24) gives

$$\text{DW} \qquad \frac{d\delta x}{dt} = A\omega e^{k(z_0+\delta z)} \cos\left(k(x_0 + \delta x) - \omega t\right) \tag{1.26a}$$

$$\text{DW} \qquad \frac{d\delta z}{dt} = A\omega e^{k(z_0+\delta z)} \sin\left(k(x_0 + \delta x) - \omega t\right) \tag{1.26b}$$

This doesn't look any simpler! But now we do a Taylor expansion, using the fact that δx and δz are small quantities. Writing out only the first few terms explicitly, equations (1.26) become

$$\frac{d\delta x}{dt} = A\omega \left[e^{kz_0}(1 + k\delta z + \cdots)\right] \left[\cos\left(kx_0 - \omega t\right) - k \sin\left(kx_0 - \omega t\right)\delta x + \cdots\right] \tag{1.27a}$$

$$\frac{d\delta z}{dt} = A\omega \left[e^{kz_0}(1 + k\delta z + \cdots)\right] \left[\sin\left(kx_0 - \omega t\right) + k \cos\left(kx_0 - \omega t\right)\delta x + \cdots\right] \tag{1.27b}$$

If we keep only the largest terms on the right-hand side, we have

$$\text{DW} \qquad \frac{d\delta x}{dt} = A\omega e^{kz_0} \cos\left(kx_0 - \omega t\right) \tag{1.28a}$$

$$\text{DW} \qquad \frac{d\delta z}{dt} = A\omega e^{kz_0} \sin\left(kx_0 - \omega t\right) \tag{1.28b}$$

Note that (1.28) also result from replacing (x, z) by (x_0, z_0) in (1.15c-d). We solve (1.28) by direct integration, obtaining

$$\text{DW} \qquad \delta x(t) = C_1 - Ae^{kz_0} \sin\left(kx_0 - \omega t\right) \tag{1.29a}$$

$$\text{DW} \qquad \delta z(t) = C_2 + Ae^{kz_0} \cos\left(kx_0 - \omega t\right) \tag{1.29b}$$

where C_1 and C_2 are constants of integration. Since $(\delta x, \delta z)$ represent the *departure* of (x_p, z_p) from their average, both C_1 and C_2 must vanish. Hence

$$\text{DW} \qquad \delta x(t) = -Ae^{kz_0} \sin\left(kx_0 - \omega t\right) \tag{1.30a}$$

$$\text{DW} \qquad \delta z(t) = Ae^{kz_0} \cos\left(kx_0 - \omega t\right) \tag{1.30b}$$

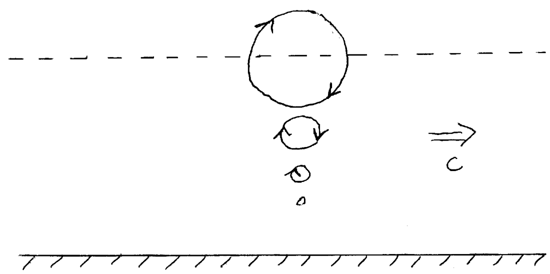

Figure 1.5: Particle paths in deep water.

Suppose $x_0 = 0$. This corresponds to choosing a fluid particle that is located directly under the crest of the wave at time $t = 0$. Since every fluid particle is located under a wave crest at some time, this is not a real restriction. In this case (1.30) reduces to

$$\text{DW} \qquad \delta x(t) = Ae^{kz_0} \sin(\omega t) \tag{1.31a}$$

$$\text{DW} \qquad \delta z(t) = Ae^{kz_0} \cos(\omega t) \tag{1.31b}$$

The trajectory corresponding to (1.31) is a circle of radius Ae^{kz_0}. This radius is largest for fluid particles whose average location is at the ocean surface ($z_0 = 0$) and decays as z_0 decreases. For a wave traveling to the right as shown in figure 1.5, the particle moves clockwise in the circle, with the top of the circle corresponding to the particle's location beneath the crest, and the bottom of the circle corresponding to its location beneath the trough. However, because $Ak \ll 1$, the particle's displacement from its average location is always much smaller than the wavelength of the wave. At the order of the approximation (1.31), the fluid particle returns, each wave period, to its location at the beginning of the period; there is no net displacement in the direction of wave propagation.

For the shallow-water wave, we see from (1.19c-d) that the vertical velocity w is smaller than the horizontal velocity u by a factor $kH \ll 1$. Thus in SW the fluid particles move back and forth in the horizontal direction, with negligible vertical displacement. For SW, we obtain instead of (1.27)

$$\text{SW} \qquad \frac{d\delta x}{dt} = \frac{A\omega}{kH}\left[\cos(kx_0 - \omega t) - k\sin(kx_0 - \omega t)\delta x + \cdots\right] \tag{1.32a}$$

$$\text{SW} \qquad \frac{d\delta z}{dt} = A\omega\left[1 + (z_0 + \delta z)/H\right]\left[\sin(kx_0 - \omega t) + k\cos(kx_0 - \omega t)\delta x + \cdots\right] \tag{1.32b}$$

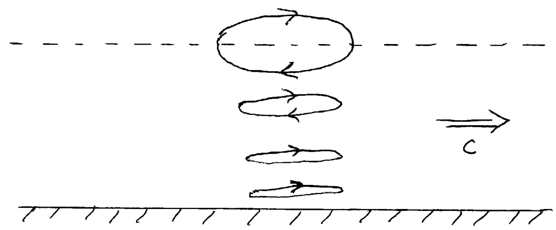

Figure 1.6: Particle paths in shallow water.

Again taking $x_0 = 0$, we have

$$\text{SW} \qquad \frac{d\delta x}{dt} = \frac{A\omega}{kH}\left[\cos(\omega t) + k\sin(\omega t)\delta x + \cdots\right] \tag{1.33a}$$

$$\text{SW} \qquad \frac{d\delta z}{dt} = A\omega\left[1 + (z_0 + \delta z)/H\right]\left[-\sin(\omega t) + k\cos(\omega t)\delta x + \cdots\right] \tag{1.33b}$$

Keeping only the largest term on the right-hand side of (1.33), we obtain the first approximation

$$\text{SW} \qquad \frac{d\delta x}{dt} = \frac{A\omega}{kH}\cos(\omega t) \tag{1.34a}$$

$$\text{SW} \qquad \frac{d\delta z}{dt} = -A\omega(1 + z_0/H)\sin(\omega t) \tag{1.34b}$$

with solution

$$\text{SW} \qquad \delta x(t) = \frac{A}{kH}\sin(\omega t) \tag{1.35a}$$

$$\text{SW} \qquad \delta z(t) = A(1 + z_0/H)\cos(\omega t) \tag{1.35b}$$

The trajectory (1.35) is an ellipse with major axis of length $2A/kH$ in the horizontal direction, and minor axis of length $2A(1 + z_0/H)$ in the vertical direction. At the ocean bottom ($z_0 = -H$), the fluid moition is purely horizontal. See figure 1.6.

As in the DW case, the fluid particle experiences no net displacement at this first order of approximation. But suppose that instead of just neglecting the δx on the right-hand side of (1.33a), we replace it by the first approximation (1.35a). We should then get a better approximation than (1.35a).

Substituting (1.35a) into the right-hand side of (1.33a) and neglecting all smaller terms, we obtain

$$\text{SW} \qquad \frac{d\delta x}{dt} = \frac{A\omega}{kH}\left[\cos(\omega t) + \frac{A}{H}\sin^2(\omega t)\right] \tag{1.36}$$

Using the identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, we integrate (1.36) to find that

$$\text{SW} \qquad \delta x(t) = \frac{A}{kH}\sin(\omega t) - \frac{A^2}{2kH^2}\sin(2\omega t) + \frac{A^2\omega}{2kH^2}t \tag{1.37}$$

Once again, (1.37) should be a better approximation to the particle displacement than (1.35a). The first term in (1.37) is the only term present in (1.35a). The second term in (1.37) is, like the first term, an oscillatory term. Like the first term, it causes no net displacement of the fluid particle. Moreover, it is much smaller than the first term, because it is proportional to the *square* of the wave amplitude A, which we have assumed to be infinitesimal.

The last term in (1.37) is also proportional to A^2, but unlike the first two terms, it is not an oscillatory term. In fact, it is *proportional* to the time t. It represents a small, steady drift—often called the *Stokes drift*—of the fluid particles, at speed

$$\text{SW} \qquad c_{drift} = \frac{A^2\omega}{2kH^2} = \frac{1}{2}\frac{A^2}{H^2}c \tag{1.38}$$

in the direction of wave propagation. Because A is infinitesimal, the drift speed c_{drift} is much smaller than the phase speed c.

For deep-water waves, we obtain the analog of (1.36) by substituting both of (1.30) into (1.27a) with $x_0 = 0$. We obtain

$$\begin{aligned}
\text{DW} \qquad \frac{d\delta x}{dt} &= A\omega\left[e^{kz_0}(1+k\delta z)\right]\left[\cos(\omega t) + k\sin(\omega t)\delta x\right] \\
&= A\omega\left[e^{kz_0}(1+Ae^{kz_0}k\cos(\omega t))\right]\left[\cos(\omega t) + Ae^{kz_0}k\sin^2(\omega t)\right] \\
&= A\omega e^{kz_0}\left[\cos(\omega t) + Ake^{kz_0}(\cos^2(\omega t)+\sin^2(\omega t))\right] \\
&= A\omega e^{kz_0}\left[\cos(\omega t) + Ake^{kz_0}\right]
\end{aligned} \tag{1.39}$$

In the next-to-last step of (1.39), we have kept the terms proportional to A and A^2, but thrown away the smaller terms of size A^3. Integrating (1.39) gives the first approximation (1.30a), plus smaller oscillatory terms, plus a drift term, just as in (1.37). To get the drift term by itself, we only need to

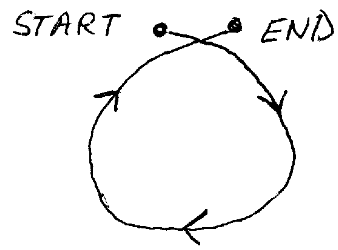

Figure 1.7: Particle path exhibiting Stokes drift.

take the time average of (1.39), obtaining the drift velocity for deep water waves:

$$\text{DW} \qquad c_{drift} = A^2 \omega k e^{2kz_0} = A^2 k^2 e^{2kz_0} c \tag{1.40}$$

At the order of the approximation (1.30), the fluid particles move in closed paths, always returning to the place where they started. At the order of the more accurate approximation (1.39), the particle paths don't quite close. The fluid particles experience a small net displacement in the direction of wave propagation, as shown in figure 1.7

The drift speed is the speed at which our piece of tissue paper moves toward the beach. In deep water, at the surface, the tissue paper drifts with the speed $A^2\omega^3/g$, according to (1.40). If the paper is neutrally buoyant at the average submerged location z_0, its drift speed is smaller by a factor of e^{2kz_0} than at the surface. The SW drift speed (1.38) is independent of z. In both DW and SW (and in the intermediate range, in which we would have to use (1.10) and (1.11) in (1.23)), the drift speed is toward the beach at all levels. If this were all that were happening, the waterline would advance steadily up the beach, eventually reaching La Jolla Shores drive. This doesn't happen! Instead the shoreward drift is balanced by a return current that tends to be independent of z. Since the drift current is smallest near the sea bottom, the return current is most apparent there, where it is sometimes referred to as *undertow*. The return current need not be uniform in the longshore direction; often it is concentrated in rip currents at particular locations along the beach. But it must exist, to compensate for the wave-induced drift of water particles toward the beach.

It is important to emphasize that our formula for the drift velocity is—like almost everything else in this course—based upon linear theory, which assumes that the wave amplitude A is small. Once again, this is the basic assumption underlying Postulates #1 and #2. This means that our drift-velocity formula is really only accurate *seaward* of the breaker zone. The breaker zone is the boundary between the seaward region of approximately *linear* dynamics in which Postulates #1 and #2 apply, and the shoreward region of highly *nonlinear* dynamics, in which a more exact physical description must be used. The physics of the shoreward region, where wave breaking leads to the formation of turbulent bores and rip currents, is much harder to analyze than the seaward region.

We have learned almost everything we can from Postulate #1 by itself. It is time to invoke Postulate #2, to see what happens when two or more waves, each satisfying Postulate #1, are present together. However, before moving on, we re-state Postulate #1 in its most general form:

Postulate #1. (re-stated, general version)
A single wave is described by the equations

$$\eta = A\cos(kx + ly - \omega t + \alpha) \tag{1.41a}$$

$$\mathbf{u} = (u, v) = A\omega \frac{\mathbf{k}}{\kappa} \frac{\cosh(\kappa(H+z))}{\sinh(\kappa H)} \cos(kx + ly - \omega t + \alpha) \tag{1.41b}$$

$$w = A\omega \frac{\sinh(\kappa(H+z))}{\sinh(\kappa H)} \cos(kx + ly - \omega t - \pi/2 + \alpha) \tag{1.41c}$$

Here, $\mathbf{u} = (u, v)$ is the horizontal fluid velocity; $\mathbf{k} = (k, l)$ is the wave vector; and the frequency ω is given by the dispersion relation

$$\omega = \sqrt{g\kappa \tanh(\kappa H)} \tag{1.41d}$$

with $\kappa = |\mathbf{k}| = \sqrt{k^2 + l^2}$. The wave propagates in the direction of $\mathbf{k}$. Note that $\mathbf{k}/\kappa$ is the unit vector in the direction of $\mathbf{k}$. The constant α represents an arbitrary phase shift.

For each wave of the form (1.41) there are four free parameters—four things that you can choose arbitrarily. These are the two components of the horizontal wavevector $\mathbf{k}$, the amplitude A, and the phase constant α. Everything else is determined. That is, the physics dictates the frequency ω, the *relative* amplitudes of the u-, v-, and w-waves and their *relative* phases.

Our previous statement of Postulate #1 corresponds to the choices $\mathbf{k} = (k, 0)$ with k positive, and $\alpha = 0$. However, these are not real restrictions.

When only a single wave is present, it is always possible to rotate the horizontal axes so that the wave propagates in the direction of the positive x-axis, and it is always possible to shift the time coordinate so that the phase constant α disappears. Therefore, our discussion of the single wave was a completely general one.

Postulate #2 allows us to add together as many waves satisfying Postulate #1 as we want. However, in the case of two or more waves, it is not possible to simplify the situation nearly as much. We can rotate the axes so that *one* of the waves propagates in the x-direction, and we can eliminate the phase constant for *one* of the waves, but the general form of (1.41) is necessary to cover all physical possibilities.

Chapter 2

Two waves

Postulate #1 tells us what basic waves are like. Postulate #2 tells us that we can add them together. This property of adding waves together is tremendously important, because, as it turns out, you can describe every possible situation by adding together a sufficient number of basic waves. In typical situations, sufficient means infinite. You need an infinite number of basic waves to describe the situation! However, adding up an infinite number of waves involves some fairly serious mathematics, which we begin to learn in the next chapter.

What happens when waves are added together, or, as wave dynamicists would prefer to say, superposed? In one word: *interference*. In some places this interference is constructive, and the waves add up to something big. In other places the interference is destructive, and the waves tend to cancel each other out. You can demonstrate interference without considering an infinite number of waves. In fact, you need only two waves to demonstrate some of the most important properties of interfering waves.

In this section we consider two examples in which two waves interfere. In the first example, the two waves are identical except for the fact that they propagate in opposite directions. The result is a standing wave—something that you have almost certainly seen before.

The second example is more interesting. It involves two *nearly* identical waves moving in the same direction. If the two waves were absolutely identical, the result would be very boring indeed—a single wave twice as large. But two nearly identical *basic* waves produce a single, *slowly varying* wave, which offers the clearest illustration of something called the group velocity. The group velocity is possibly the most important tool for understanding

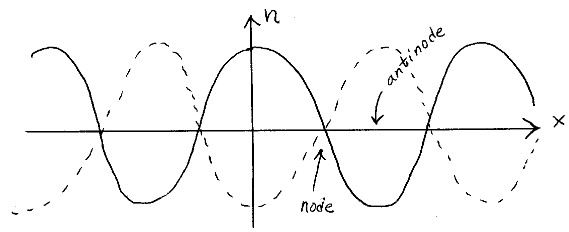

Figure 2.1: Standing wave at two times separated by a half period.

ocean waves. It will take several lectures to explain its full importance.

Now for the first example. Consider the case of two waves, with the same frequency and amplitude, but propagating in opposite directions. The surface elevation is

$$\eta = A\cos(kx - \omega t) + A\cos(kx + \omega t) \tag{2.1}$$

where k and ω are positive constants that satisfy the dispersion relation (1.2). By the trigonometric identity

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \tag{2.2}$$

(2.1) is equivalent to

$$\eta = 2A\cos(kx)\cos(\omega t) \tag{2.3}$$

Interference between the waves has produced a *standing wave*. The *nodes*, located where $kx = \pi/2 + n\pi$ for n an integer, are the places where the surface elevation always vanishes. At the *antinodes*, $kx = n\pi$, the surface elevation oscillates between $+2A$ and $-2A$, as shown in figure 2.1.

By Postulate #1, the horizontal velocity component corresponding to (2.1) is

$$u = A\omega\frac{\cosh(k(H + z))}{\sinh(kH)}\cos(kx-\omega t) - A\omega\frac{\cosh(k(H + z))}{\sinh(kH)}\cos(kx+\omega t) \tag{2.4}$$

By (2.2), this is

$$u = 2A\omega\frac{\cosh(k(H + z))}{\sinh(kH)}\sin(kx)\sin(\omega t) \tag{2.5}$$

The nodes of u are at $kx = n\pi$, and the antinodes of u are at $kx = \pi/2 + n\pi$.

What are these standing waves good for? The solution (2.3) and (2.5) describes a progressive wave with wavelength $2\pi/k$ that is perfectly reflected from a seawall located at $x = 0$ (or at any $x = n\pi/k$). Since the seawall is a rigid vertical barrier, we must have $u = 0$ at the seawall. This solution works to either side of the seawall. The incoming wave can have any wavelength; the outgoing wave will have the same wavelength.

Now suppose there are *two* seawalls located at $x = 0$ and $x = L$. Alternatively, suppose we have a harbor or bay, located on $0 < x < L$, with vertical sidewalls at $x = 0$ and $x = L$. Since there can be no flow through the sides of the bay, the waves must obey the *boundary condition* $u = 0$ at $x = 0$ and at $x = L$, at all times. In this case the wavenumber k is no longer arbitrary. The velocity (2.5) satisfies both boundary conditions only if k belongs to the infinite discrete set

$$\{k_n = \frac{n\pi}{L}\} = \{\frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots\} \tag{2.6}$$

This choice of wavenumber corresponds to placing the nodes of u at $x = 0$ and $x = L$.

For each such wavenumber in the set (2.6), we can choose a different arbitrary amplitude A, but the frequency ω is determined by the dispersion relation. Thus a possible solution in the bay is

$$\eta = 2A_n \cos(k_n x)\cos(\omega_n t) \tag{2.7a}$$

$$u = 2A_n\omega_n \frac{\cosh(k_n(H+z))}{\sinh(k_n H)} \sin(k_n x)\sin(\omega_n t) \tag{2.7b}$$

where A_n is any number, n is any positive integer, $k_n = n\pi/L$, and

$$\omega_n = \sqrt{gk_n \tanh(k_n H)} \tag{2.8}$$

The solution (2.7) satisfies our two postulates, and it satisfies the boundary condition at each end of the bay. It is called a *normal mode*.

Postulate #2 allows us to add together as many normal modes—as many pairs of basic waves—of the form (2.7) as we wish. The result is

$$\eta = \sum_{n=1}^{\infty} 2A_n \cos(k_n x)\cos(\omega_n t) \tag{2.9a}$$

$$u = \sum_{n=1}^{\infty} 2A_n\omega_n \frac{\cosh(k_n(H+z))}{\sinh(k_n H)} \sin(k_n x)\sin(\omega_n t) \tag{2.9b}$$

which is the sum of the normal modes in a one-dimensional bay. The horizontal velocity corresponding to mode n has $n-1$ nodes within the bay itself, besides the nodes at each end.

Now what about the y-direction? (When was the last time you saw a one-dimensional bay?) Suppose that the other two boundaries lie at $y = 0, d$. The boundary conditions there are $v = 0$; the y-component of velocity must vanish. But our solution satisfies these boundary conditions automatically, because our solution has $v = 0$ *everywhere*. It describes a mode in which the water particles move only in the x-direction. Of course, we can find analogous modes in which the water moves only in the y-direction. And there are even modes that oscillate simultaneously in *both* directions. For the moment, we restrict ourselves to the one-dimensional case.

In the shallow-water limit, the complete, one-dimensional, normal-mode solution is

$$\eta = \sum_{n=1}^{\infty} 2A_n \cos(k_n x) \cos(\omega_n t + \alpha_n) \tag{2.10}$$

and

$$\text{SW} \qquad u = \sum_{n=1}^{\infty} \frac{2A_n \omega_n}{k_n H} \sin(k_n x) \sin(\omega_n t + \alpha_n) \tag{2.11}$$

with

$$\text{SW} \qquad \omega_n = \sqrt{gH} k_n \tag{2.12}$$

We have added an arbitrary phase constant to each mode. Equations (2.10-12) represent the *general solution* for waves in a one-dimensional bay. What exactly does this mean? It means that every possibility allowed by the physics corresponds to (2.10-12) for some choice of the A_n's and α_n's. To see how this works, we suppose that u and η are given at some initial time, say $t = 0$. We will show that knowledge of $u(x, 0)$ and $\eta(x, 0)$ determines all the A_n's and all the α_n's. The solution at all subsequent times is then given by (2.10) and (2.11).

Setting $t = 0$ in (2.10-11), and making use of (2.12), we obtain

$$\eta(x, 0) = \sum_{n=1}^{\infty} 2A_n \cos(k_n x) \cos(\alpha_n) \tag{2.13}$$

and

$$\sqrt{\frac{H}{g}} u(x, 0) = \sum_{n=1}^{\infty} 2A_n \sin(k_n x) \sin(\alpha_n) \tag{2.14}$$

Our aim is to determine the A_n's and α_n's from (2.13) and (2.14). To do this we multiply both sides of (2.13) by $\cos(k_m x)$ where $k_m = m\pi/L$, and integrate the resulting equation between $x = 0$ and $x = L$. The result is

$$\int_0^L dx\, \eta(x,0)\cos(k_m x) = \sum_{n=1}^{\infty} 2A_n \cos(\alpha_n) \int_0^L dx\ \cos(k_n x)\cos(k_m x) \tag{2.15}$$

However,

$$\int_0^L dx\ \cos(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x) = \begin{cases} L/2 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

Hence only one of the terms in the sum on the right-hand side of (2.15) is nonzero, and (2.15) reduces to

$$\int_0^L dx\, \eta(x,0)\cos(k_m x) = LA_m \cos(\alpha_m) \tag{2.17}$$

which holds for every positive integer m. We obtain a similar equation by multiplying (2.14) by $\sin(k_m x)$ and integrating over the same interval to get

$$\sqrt{\frac{H}{g}} \int_0^L dx\, u(x,0)\sin(k_m x) = LA_m \sin(\alpha_m) \tag{2.18}$$

For each m, equations (2.17) and (2.18) determine A_m and α_m. This proves the claim that, for arbitrary initial conditions $u(x,0)$ and $\eta(x,0)$, the wave field is uniquely determined as the sum of normal modes. The general method of representing an unknown field as the sum of sines and cosines, and determining the amplitudes and phases using tricks like (2.16), is called *Fourier analysis*. Fourier analysis is the primary mathematical tool for studying linear waves.

Now for the second example. Consider the case of two waves, again with the same amplitude, but now propagating in the *same* direction, toward positive x:

$$\eta = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t) \tag{2.19}$$

Each wave obeys the dispersion relation (1.2). Defining

$$\bar{k} = \frac{1}{2}(k_1 + k_2), \quad \Delta k = \frac{1}{2}(k_2 - k_1), \quad \bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2), \quad \Delta\omega = \frac{1}{2}(\omega_2 - \omega_1) \tag{2.20}$$

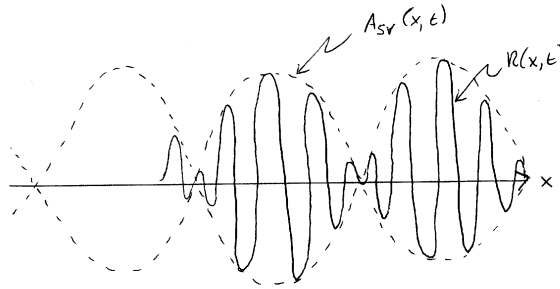

Figure 2.2: The envelope A_{SV} moves at half the speed of η.

we can rewrite (2.19) as

$$\eta = A\cos(\bar{\theta} - \Delta\theta) + A\cos(\bar{\theta} + \Delta\theta) \tag{2.21}$$

where

$$\bar{\theta} = \bar{k}x - \bar{\omega}t \quad \text{and} \quad \Delta\theta = \Delta k\, x - \Delta\omega\, t \tag{2.22}$$

By the same trigonometric identity (2.2), (2.21) is equivalent to

$$\eta = 2A\cos(\Delta\theta)\cos(\bar{\theta}) = 2A\cos(\Delta k\, x - \Delta\omega\, t)\cos(\bar{k}x - \bar{\omega}t) \tag{2.23}$$

Suppose that the two wavenumbers, and hence the two frequencies, differ by only a small amount. For small Δk and $\Delta\omega$, (2.23) describes a basic wave with wavenumber $\bar{k}$, frequency $\bar{\omega}$, and *slowly varying amplitude*

$$A_{SV}(x,t) \equiv 2A\cos(\Delta k\, x - \Delta\omega\, t) \tag{2.24}$$

By 'slowly varying' we mean that A_{SV} does not change by much over a wavelength $2\pi/\bar{k}$ or wave period $2\pi/\bar{\omega}$ of the 'carrier wave.' The expression (2.24) is also called the wave *envelope*. Refer to figure 2.2. The wave crests and troughs move at the phase speed

$$\bar{c} = \frac{\bar{\omega}}{\bar{k}} \tag{2.25}$$

but the envelope moves at the *group velocity*

$$c_g = \frac{\Delta\omega}{\Delta k} \tag{2.26}$$

In the limit of very small differences between the two wavenumbers and frequencies, these equations become

$$c = \frac{\omega}{k} \quad \text{and} \quad c_g = \frac{d\omega}{dk} \tag{2.27}$$

This result holds for any type of wave, that is, for any dispersion relation $\omega = \omega(k)$. In the case of deep water waves, we have

$$\text{DW} \qquad \omega = \sqrt{gk} \tag{2.28}$$

Therefore

$$\text{DW} \qquad c = \sqrt{\frac{g}{k}} \quad \text{and} \quad c_g = \frac{d}{dk}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c \tag{2.29}$$

In deep water, the envelope moves at half the speed of the crests and troughs. Individual waves appear to pass through the group, from rear to front, growing and then decaying in turn. In the shallow water case, we have

$$\text{SW} \qquad \omega = \sqrt{gH}k \tag{2.30}$$

so

$$\text{SW} \qquad c = \sqrt{gH} \quad \text{and} \quad c_g = \frac{d}{dk}(\sqrt{gH}k) = \sqrt{gH} = c \tag{2.31}$$

In the shallow water case, the phase velocity and the group velocity are the same; the wave crests and troughs hold their place within the group. Waves in which the phase speed does *not* depend on the wavenumber (i.e. does not depend on the wavelength) are said to be *nondispersive*. For nondispersive waves, the group velocity always equals the phase velocity.

If the dispersion relation is such that the phase velocity depends on the wavenumber (as in the deep water case), then the group velocity differs from the phase velocity. However, the group velocity can be greater or less than the phase velocity, depending on the particular dispersion relation. For deep water gravity waves, the phase velocity exceeds the group velocity, but for capillary waves the opposite is true (as you will see in a homework problem).

Wave groups resemble what surfers call *sets*, short series of high amplitude waves followed by a series of smaller waves. Single groups—called wave packets—can also occur (figure 2.3), and it is found that these packets, like the envelopes in our example, move at the group velocity. Once again, in DW,

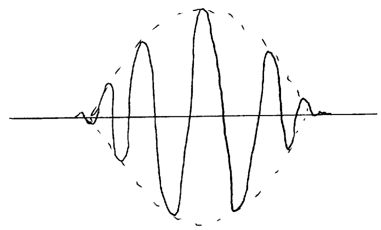

Figure 2.3: A wavepacket.

the individual wave crests and troughs move faster than the group velocity, and thus appear from nowhere at the back of the wave packet, and disappear into nothing at its front.

At first sight, the sudden appearance and equally sudden vanishing of individual waves seems disturbing. It seems to violate some conservation law. But individual waves are not generally conserved. *Energy* is conserved. And energy is associated with the envelopes—not with the individual waves. Thus energy moves at the group velocity—not the phase velocity.

What is the energy associated with the waves described by (2.23)? First there is the potential energy associated with height of the water particles in the Earth's gravity field. Then there is the kinetic energy associated with the velocity of the fluid particles. Let us try to calculate the latter. To do this, we need to write down the velocity field associated with (2.23). By Postulate #1, and assuming we are in deep water, it is

$$\text{DW} \qquad u = 2A\omega e^{\bar{k}z}\cos(\Delta k\, x - \Delta\omega\, t)\cos(\bar{k}x - \bar{\omega}t) \tag{2.32}$$

plus a similar expression for the vertical velocity w. The kinetic energy per unit volume is $\frac{1}{2}\rho(u^2 + w^2)$, where ρ is the mass density, the mass per unit volume. Hence the average kinetic energy per unit horizontal area, at $x = x_0$ and $t = t_0$, is

$$\text{DW} \qquad K(x_0, t_0) = \rho \int_{-\infty}^{0} dz\, \frac{1}{\lambda} \int_{x_0}^{x_0+\lambda} dx\, \frac{1}{2}(u^2 + w^2) \tag{2.33}$$

By *average*, we mean the average over a wavelength; that explains the second integral in (2.33).

Let's calculate the contribution of the horizontal velocity u to (2.33). To do this we must integrate

$$\begin{aligned}
&\int_{-\infty}^{0} dz\, \frac{1}{\lambda} \int_{x_0}^{x_0+\lambda} \frac{1}{2}\, 4A^2\omega^2 e^{2\bar{k}z} \cos^2(\Delta k\; x - \Delta\omega\; t_0) \cos^2(\bar{k}x - \bar{\omega}t_0) \\
&\approx 2A^2\omega^2 \cos^2(\Delta k\; x_0 - \Delta\omega\; t_0) \int_{-\infty}^{0} dz\; e^{2\bar{k}z} \frac{1}{\lambda} \int_{x_0}^{x_0+\lambda} dx\; \cos^2(\bar{k}x - \bar{\omega}t_0) \\
&= 2A^2\omega^2 \cos^2(\Delta k\; x_0 - \Delta\omega\; t_0) \frac{1}{2\bar{k}} \frac{1}{2} \\
&= \frac{1}{8} \frac{\omega^2}{\bar{k}} 4A^2 \cos^2(\Delta k\; x_0 - \Delta\omega\; t_0) \\
&= \frac{1}{8} \frac{\omega^2}{\bar{k}} A_{SV}(x_0, t_0)^2
\end{aligned} \tag{2.34}$$

where $A_{SV}(x_0, t_0)$ is the slowly varying amplitude, defined by (2.24). In the second step of (2.34) we have assumed that the factor $\cos(\Delta k\; x - \Delta\omega\; t_0)$ changes so little over a single wavelength that we can replace it by its constant value at (x_0, t_0). To compute the total average kinetic energy, we must compute the contribution of the vertical velocity w. It turns out that this is exactly equal to (2.34). Therefore the total kinetic energy is

$$\text{DW} \qquad K = \rho \frac{1}{4} \frac{\omega^2}{\bar{k}} {A_{SV}}^2 = \frac{1}{4} \rho g\; {A_{SV}}^2 \tag{2.35}$$

Next we need to compute the average potential energy. This, it turns out, is exactly equal to the kinetic energy! So the total average energy per unit horizontal area is

$$E(x,t) = \frac{1}{2} \rho g\; A_{SV}(x,t)^2 \tag{2.36}$$

It turns out that (2.36) is correct for both deep water and for shallow water waves. In fact, it holds in the general case of arbitrary depth. The important fact for us is that (2.36) is proportional to the *square* of the slowly varying amplitude $A_{SV}(x,t)$. It is solely determined by the wave envelope, which moves at the group velocity. This justifies the statement that the energy moves at the group velocity.

Let $x_1 < x_2$ be two fixed locations. If the energy between x_1 and x_2 is conserved, then

$$\frac{d}{dt} \int_{x_1}^{x_2} dx\; E(x,t) = F(x_1,t) - F(x_2,t) \tag{2.37}$$

where $F(x,t)$ is the flux of energy *past* x, in the positive x-direction, at time t. Equation (2.37) holds if there is no local forcing or dissipation. If (2.37) holds for any x_1 and x_2, then it must be true that

$$\frac{\partial E}{\partial t} = -\frac{\partial F}{\partial x} \tag{2.38}$$

Equations (2.37) and (2.38) are equivalent. In physics, equations of the form (2.38) are called *conservation laws.*

Our statement that energy moves at the group velocity is the same as saying

$$F = c_g E \tag{2.39}$$

Then (2.38) becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(c_g E) = 0 \tag{2.40}$$

Equation (2.40) expresses the conservation of wave energy in one dimension. It is one of the most important equations in the study of waves.

The energy conservation law (2.40) allows us to predict how waves change in amplitude as they approach the beach. Earlier we showed how the slowly varying dispersion relation

$$\text{SW} \qquad \omega = \sqrt{gH(x)}k(x) \tag{2.41}$$

allows us to predict the change in wavenumber $k(x)$ that occurs in a wave directly incident on a beach. From (2.41) we found that the wavenumber varies as

$$\text{SW} \qquad k(x) \propto \frac{1}{\sqrt{H(x)}} \tag{2.42}$$

Equation (2.40) allows us to make the corresponding prediction for the slowly varying wave amplitude $A(x)$. Outside the breaker zone, wave dissipation is negligible, so (2.40) holds. In the case of steady waves, $\partial E/\partial t = 0$, and (2.40) implies

$$c_g \; E = \text{constant} \tag{2.43}$$

For shallow water waves, this is implies

$$\text{SW} \qquad \sqrt{gH(x)} \; A^2(x) = \text{constant} \tag{2.44}$$

so the wave amplitude varies as

$$\text{SW} \qquad A(x) \propto \frac{1}{H(x)^{1/4}} \tag{2.45}$$

The increase in amplitude that occurs as waves shoal leads to wave breaking. According to one idea, shoaling waves break when

$$2A = 0.78\ H \tag{2.46}$$

Chapter 3

Many waves

In the previous lecture, we considered the case of two basic waves propagating in one horizontal dimension. However, Postulate #2 lets us have as many basic waves as we want. Suppose we want to have N waves. If N waves are present, the surface elevation takes the form

$$\eta(x,t) = \sum_{i=1}^{N} \hat{A}_i \cos(k_i x - \omega(k_i)t + \alpha_i) \tag{3.1}$$

where the $\hat{A}_i$, k_i, and α_i represent $3N$ arbitrary parameters. We put the hats on $\hat{A}_i$ so we can use un-hatted A_i for something else. As always, the $\omega(k_i)$ are determined by the dispersion relation,

$$\omega(k_i) = \sqrt{g k_i \tanh(k_i H)} \tag{3.2}$$

and are always positive. If we define

$$\theta_i = k_i x - \omega(k_i)t \tag{3.3}$$

then (3.1) can be written in the form

$$\eta(x,t) = \sum_{i=1}^{N} \hat{A}_i \cos(\theta_i + \alpha_i) = \sum_{i=1}^{N} (\hat{A}_i \cos\alpha_i \cos\theta_i - \hat{A}_i \sin\alpha_i \sin\theta_i) \tag{3.4}$$

Then, defining

$$A_i = \hat{A}_i \cos\alpha_i \quad \text{and} \quad B_i = -\hat{A}_i \sin\alpha_i \tag{3.5}$$

we have

$$\eta(x,t) = \sum_{i=1}^{N} A_i \cos(k_i x - \omega(k_i)t) + B_i \sin(k_i x - \omega(k_i)t) \tag{3.6}$$

The form (3.6) is equivalent to (3.1), but it is sometimes more useful. The arbitrary parameters in (3.6) are A_i, B_i, and k_i. In both (3.1) and (3.6), the sign of each k_i determines the propagation direction of the corresponding wave.

If you have studied mechanics, then you know that the evolution of any mechanical system is determined by:

1. Newton's laws of motion, and
2. the location and velocity of every component of the system at some initial time, say $t = 0$.

The corresponding statement for us is that $\eta(x,t)$ is determined by:

1. Postulates #1 and #2, and
2. the *initial conditions* $\eta(x, 0)$ and $\partial\eta/\partial t(x, 0)$.

Suppose the latter are given to be $F(x)$ and $G(x)$ respectively. Then the solution of the initial value problem corresponds to choosing the arbitrary parameters in (3.6) to satisfy

$$\eta(x,0) = F(x) \quad \text{and} \quad \frac{\partial \eta}{\partial t}(x,0) = G(x) \tag{3.7}$$

We can do this with N waves if we can choose the wavenumbers and wave amplitudes to satisfy the equations

$$F(x) = \sum_{i=1}^{N} A_i \cos(k_i x) + B_i \sin(k_i x) \tag{3.8}$$

$$G(x) = \sum_{i=1}^{N} A_i \omega(k_i) \sin(k_i x) - B_i \omega(k_i) \cos(k_i x) \tag{3.9}$$

Is it ever possible to do this? If so, how many waves are needed?

Well, you might get lucky. For particularly simple $F(x)$ and $G(x)$, it might turn out that you need only a few waves. For example, if $F(x) = 5\cos(3x)$ and $G(x) = 5\omega(3)\sin(3x)$, then you need only a single wave. However, we are interested in the general case, in which $F(x)$ and $G(x)$ are completely arbitrary; they can be anything whatsoever. It turns out that, in the general case, you need every possible wavenumber. Thus, the general solution takes the form

$$\eta(x,t) = \int_{-\infty}^{+\infty} dk\ [A(k)\cos(kx - \omega(k)t) + B(k)\sin(kx - \omega(k)t)] \tag{3.10}$$

in which the sum in (3.6) has been replaced by an integral. The quantities $A(k)dk$ and $B(k)dk$ in (3.10) are analogous to the quantities $A(k_i)$ and $B(k_i)$ in (3.6). The initial value problem (3.7) is solved if we can find functions $A(k)$ and $B(k)$ that satisfy

$$F(x) = \int_{-\infty}^{+\infty} dk\ [A(k)\cos(kx) + B(k)\sin(kx)] \tag{3.11}$$

$$G(x) = \int_{-\infty}^{+\infty} dk\ [A(k)\omega(k)\sin(kx) - B(k)\omega(k)\cos(kx)] \tag{3.12}$$

To find these $A(k)$ and $B(k)$, we make use of a very powerful theorem in mathematics.

Fourier's theorem. For almost any function $f(x)$,

$$f(x) = \int_{0}^{\infty} dk\ [a(k)\cos(kx) + b(k)\sin(kx)] \tag{3.13}$$

where

$$a(k) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx\ f(x)\cos(kx) \tag{3.14a}$$

$$b(k) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx\ f(x)\sin(kx) \tag{3.14b}$$

for all positive k.

In essence, Fourier's theorem says that you can express any function $f(x)$ as the sum of sines and cosines, provided that you assign the right weight to each sine and each cosine; (3.14) tells you how to assign the weights. If you are given $f(x)$, then $a(k)$ and $b(k)$ are determined by (3.14). Conversely, if you are given $a(k)$ and $b(k)$, then $f(x)$ is determined by (3.13). The functions $a(k)$ and $b(k)$ are said to be the Fourier transform of $f(x)$. The function $f(x)$ is the *inverse* transform of $a(k)$ and $b(k)$. The transform and its inverse transform constitute a *Fourier transform pair*. Either member of the pair determines the other member. You might be wondering why the transform contains two functions, $a(k)$ and $b(k)$, while $f(x)$ is only one function. That seems unfair! The reason is that $a(k)$ and $b(k)$ are defined only for positive k, while $f(x)$ is defined for both positive and negative x.

This is a good place to say that Fourier's theorem can be stated in a great many equivalent but somewhat dissimilar forms. For example, instead of (3.13) many books write

$$f(x) = \int_{-\infty}^{\infty} dk\ \hat{a}(k) e^{ikx} \tag{3.15}$$

where $\hat{a}(k)$ is complex. In fact, if you look up Fourier's theorem in a math book, you are more likely to find (3.15) than (3.13). In this course, we will use (3.13-14) and we will not confuse things by discussing other forms. Your math courses will teach you all the delicate points about Fourier analysis. The goal here is to develop your physical intuition.

We shall "prove" Fourier's theorem by showing that it holds in one particular case, and then invite you to test any other cases that you like. Our test case will be the function

$$f(x) = e^{-\beta x^2} \tag{3.16}$$

where β is a positive constant. First we use (3.14) to calculate the weights. We find that

$$a(k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx\ e^{-\beta x^2} \cos(kx) = \frac{2}{\pi} \int_{0}^{+\infty} dx\ e^{-\beta x^2} \cos(kx) = \frac{1}{\sqrt{\pi\beta}} e^{-k^2/4\beta} \tag{3.17}$$

$$b(k) = 0 \tag{3.18}$$

In working out (3.17) we have used the general formula

$$\int_{0}^{\infty} e^{-qx^2} \cos(px) dx = \frac{1}{2}\sqrt{\frac{\pi}{q}} e^{-p^2/4q} \tag{3.19}$$

which can be found in any table of definite integrals. The result (3.18) follows quickly from the fact that the product of an even function and an odd function is an odd function.

If Fourier's theorem is correct, then it must by (3.13) be true that

$$e^{-\beta x^2} = \frac{1}{\sqrt{\pi\beta}} \int_0^\infty dk\; e^{-k^2/4\beta} \cos(kx) \tag{3.20}$$

We leave it to you to verify (3.20)—by using the formula (3.19) again.

So what is all this good for? We can use Fourier's theorem to obtain the general solution of our initial value problem. Recall that the problem was to find the wave amplitudes $A(k)$ and $B(k)$ that satisfy the initial conditions (3.11) and (3.12). To make (3.11) resemble (3.13), we rewrite (3.11) in the form

$$F(x) = \int_0^{+\infty} dk\; [(A(k) + A(-k)) \cos(kx) + (B(k) - B(-k)) \sin(kx)] \tag{3.21}$$

We have changed the integration limits in (3.11) to match those in (3.13). Then Fourier's theorem tells us that

$$A(k) + A(-k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx\; F(x) \cos(kx) \tag{3.22a}$$

$$B(k) - B(-k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx\; F(x) \sin(kx) \tag{3.22b}$$

Equations (3.22) are two equations in four unknowns. The four unknowns are the amplitudes $A(k)$ and $A(-k)$ of the right- and left-propagating cosine waves, and the amplitudes $B(k)$ and $B(-k)$ of the right- and left-propagating sine waves. We get two more equations for the same four unknowns by applying Fourier's theorem to our other initial condition (3.12). The result is

$$A(k) - A(-k) = \frac{1}{\pi\omega(k)} \int_{-\infty}^{+\infty} dx\; G(x) \sin(kx) \tag{3.23a}$$

$$B(k) + B(-k) = -\frac{1}{\pi\omega(k)} \int_{-\infty}^{+\infty} dx\; G(x) \cos(kx) \tag{3.23b}$$

Solving (3.22) and (3.23) for the four unknowns, we obtain

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\ F(x)\cos(kx) + \frac{1}{2\pi\omega(k)}\int_{-\infty}^{+\infty} dx\ G(x)\sin(kx) \quad (3.24a)$$

$$B(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\ F(x)\sin(kx) - \frac{1}{2\pi\omega(k)}\int_{-\infty}^{+\infty} dx\ G(x)\cos(kx) \quad (3.24b)$$

which hold for both positive and negative k. In summary, the wave field corresponding to the initial conditions (3.7) is (3.10) with the wave amplitudes given by (3.24).

This is a remarkable achievement. Using only our two postulates, we have shown how to find the wave field that results from any set of initial conditions. The result is a tribute to the power of Postulate #2, which allows us to add together as many waves satisfying Postulate #1 as we please. By adding together an infinite number of waves, we acquire the ability to handle the general case. Of course, (3.10) only gives us the surface elevation. But Postulate #1 tells us that the accompanying velocity field must be

$$u(x,t) = \int_{-\infty}^{+\infty} dk\ [A(k)\omega(k)\frac{\cosh k(H+z)}{\sinh kH}\cos(kx-\omega(k)t) + B(k)\omega(k)\frac{\cosh k(H+z)}{\sinh kH}\sin(kx-\omega(k)t)] \quad (3.25)$$

The amplitudes in (3.25) are the same ones as in (3.10)—the amplitudes given by (3.24).

There is a catch to this, as you may already be suspecting. The integrals in (3.24) and (3.25) might be very hard to do. Of course one could get lucky. For a particular $F(x)$ and $G(x)$ it might turn out that the integrals are easy to do, or to look up. More typically (3.24) are easy, but the integrals (3.10) and (3.25) are impossible. Even more typically, all the integrals are impossible to do exactly. But that is not really the point. Just by writing down (3.10), (3.24) and (3.25), we have solved the problem *in principle.* If we absolutely need a quantitatively accurate answer, we can always evaluate these integrals with the help of a computer, using numerical techniques. Sometimes, however, we don't need a perfectly accurate answer; we just want to see what's going on. What are these formulas really telling us? In that case, a good method is to consider special cases for which the calculations are easier.

In that spirit, we suppose that the initial surface elevation is a 'motionless hump.' That is, we suppose that $G(x) = 0$. We further suppose that the hump is symmetric about $x = 0$, i.e. that $F(x) = F(-x)$. An example of such a function is $F(x) = e^{-\beta x^2}$, the same function we used to illustrate Fourier's theorem. With these restrictions on the initial conditions, the wave amplitudes (3.24) take the simple form

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\; F(x)\cos(kx) \quad \text{and} \quad B(k) = 0 \tag{3.26}$$

so our solution is

$$\eta(x,t) = \int_{-\infty}^{+\infty} dk\; A(k)\cos(kx - \omega(k)t) \tag{3.27}$$

with $A(k)$ given by (3.26). The solution (3.27) contains no sine waves, and, because (3.26) tells us that $A(k) = A(-k)$, the amplitude of the left-moving wave equals the amplitude of the right-moving wave at the same wavelength. Let us rewrite (3.27) to emphasize that fact. We write

$$\eta(x,t) = \eta_L(x,t) + \eta_R(x,t) \tag{3.28}$$

where

$$\eta_R(x,t) = \int_0^{\infty} dk\; A(k)\cos(kx - \omega(k)t) \tag{3.29}$$

represents the right-moving wave, and where

$$\begin{aligned}\eta_L(x,t) &= \int_{-\infty}^{0} dk\; A(k)\cos(kx - \omega(k)t) \\ &= \int_0^{\infty} dk\; A(-k)\cos(-kx - \omega(-k)t) \\ &= \int_0^{\infty} dk\; A(k)\cos(kx + \omega(k)t)\end{aligned} \tag{3.30}$$

represents the left-moving wave. In simplifying (3.30), we have used the facts that the cosine is an even function, $\cos(s) = \cos(-s)$, and that the frequency is always positive, $\omega(k) = \omega(-k) > 0$. By the even-ness of the cosine we see that

$$\eta_L(-x,t) = \eta_R(x,t) \tag{3.31}$$

Thus the left-moving waves are a mirror image of the right-moving waves. Of course, this is a consequence of the symmetry of our initial conditions.

Our results hold for any kind of wave—deep water waves, shallow water waves, or waves in between. But suppose we are dealing with shallow water waves. This is subtle, because the integrations in (3.29) and (3.30) run over *all* k, and high enough k will certainly violate $kH \ll 1$. Their wavelengths will be less than the fluid depth. What we are really assuming is that $A(k)$ is small for those large k. And that, in turn, is an assumption about our initial conditions. Our 'motionless hump' must be very broad.

In the case of shallow water waves (3.27) becomes

$$\text{SW} \qquad \begin{aligned} \eta(x,t) &= \eta_L(x,t) + \eta_R(x,t) \\ &= \int_0^\infty dk\ A(k)\cos(k(x+ct)) + \int_0^\infty dk\ A(k)\cos(k(x-ct)) \\ &= \frac{1}{2}F(x+ct) + \frac{1}{2}F(x-ct) \end{aligned} \tag{3.32}$$

where $c = \sqrt{gH}$ is the shallow water phase speed, because by (3.21)

$$F(\xi) = 2\int_0^\infty dk\ A(k)\cos(k\xi). \tag{3.33}$$

For the particular initial condition

$$\eta(x,0) = F(x) = e^{-\beta x^2} \tag{3.34}$$

the shallow-water solution is

$$\text{SW} \qquad \eta(x,t) = \frac{1}{2}e^{-\beta(x+ct)^2} + \frac{1}{2}e^{-\beta(x-ct)^2} \tag{3.35}$$

Thus the initial hump splits symmetrically into two, mirror-image parts, which move apart without changing their shape. This property of 'not changing shape' is a peculiar property of nondispersive waves. It depends critically on the fact that all the basic cosine waves move at the same speed, regardless of wavelength. Thus it applies only to shallow water waves. For the more interesting case of deep water waves, no simplification like (3.32) is possible; we must attack the general solution (3.29-30) by other means. That is our next assignment.

Chapter 4

Waves generated by a distant storm

Imagine the following situation. Far across the Pacific, a powerful storm occurs within a specific region and a specific time interval. The wind churns up the waves, and they begin to propagate toward San Diego. Idealizing to the one-dimensional case, we let the location of the storm be (near) $x = 0$, and we let the time of the storm be (around) $t = 0$. The precise way in which the storm generates waves is quite complicated. We shall say more about it in the next chapter. For present purposes, we imagine that the storm kicks up the waves near $(x, t) = (0, 0)$, and then *stops*. We pick up the problem at that point. Thus the problem to be solved is this: given the sea state just after the storm (as initial condition), find the sea state at San Diego at a much later time, when the waves from the storm finally reach our shore.

The 'initial condition' in the aftermath of the storm will be quite complicated, but we shall idealize it as a 'motionless hump' with the same spatial symmetry as considered in the previous lecture. Then the solution to the problem is given by (3.28-30) with the amplitudes $A(k)$ determined by the exact form of the initial hump. To get the answer, we need only evaluate the integrals in (3.29) and (3.30).

The problem we have set is a severely idealized one, but it can teach us a great deal about the general situation. The key assumption—the assumption that allows us to work out the answer—is that the storm takes place at a specific place and time, and one that is well separated from the location and time at which we want to know the answer.

Suppose that, in our one-dimensional world, San Diego lies to the right of the storm, at large positive x. Then we need only keep track of the right-moving waves. We need only evaluate

$$\eta_R(x,t) = \int_0^\infty dk\ A(k)\cos(kx - \omega(k)t) \tag{4.1}$$

at large positive x and t—large t, because the waves generated at $t = 0$ take a long time to cross the ocean. With this in mind, we re-write (4.1) in the form

$$\eta_R(x,t) = \int_0^\infty dk\ A(k)\cos(t\phi(k)) \tag{4.2}$$

where

$$\phi(k) \equiv k\frac{x}{t} - \omega(k) \tag{4.3}$$

We shall evaluate (4.2) for $t \to \infty$, but with the quotient x/t in (4.3) *held fixed*. In that case, $\phi(k)$ depends only on k. This is the mathematically cleanest way to proceed. Of course x/t can have any value we want, and, by allowing x/t to take all possible values *after* we have performed the integral (4.2), we will have the solution for *all* large x and t.

What do we use for $A(k)$? The amplitudes $A(k)$ are determined by the initial conditions—the sea surface elevation just after the storm—via Fourier's theorem. We shall assume only that $A(k)$ depends *smoothly* on k. This turns out to be a critical assumption, and it deserves further comment. But the comment will make better sense *after* we have finished the calculation. For now we simply emphasize: $A(k)$ depends *smoothly* on k.

For large enough t, even small changes in $\phi(k)$ will cause rapid oscillations in the factor $\cos(t\phi(k))$ as k increases inside the integral (4.2). However, if $A(k)$ is smooth, these oscillations produce canceling contributions to (4.2). This is true everywhere except where $\phi'(k) = 0$; there changes in k produce *no* change in $\phi(k)$, as illustrated in figure 4.1. We therefore assume that, as $t \to \infty$, the dominant contribution to (4.2) comes from wavenumbers near the wavenumber k_0 at which

$$\phi'(k_0) = \frac{x}{t} - \omega'(k_0) = 0 \tag{4.4}$$

According to (4.4), k_0 is the wavenumber of the wave whose group velocity c_g satisfies $x = c_g t$.

With k_0 defined by (4.4) for our chosen x/t, we approximate

$$\eta_R(x,t) \approx \int_{k_0-\Delta k}^{k_0+\Delta k} dk\ A(k)\cos(t\phi(k)) \approx A(k_0)\int_{k_0-\Delta k}^{k_0+\Delta k} dk\ \cos(t\phi(k)) \tag{4.5}$$

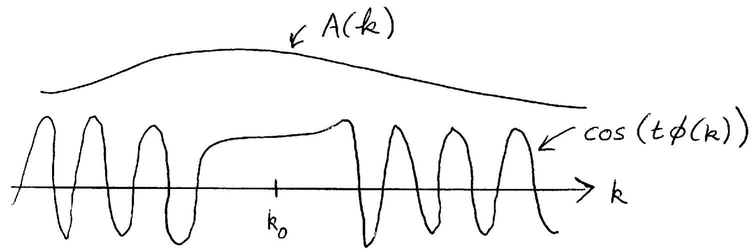

Figure 4.1: The two factors in the integrand of eqn (4.2).

where Δk is a small, arbitrary constant. In (4.5) we have replaced $A(k)$ by $A(k_0)$ because, assuming that $A(k)$ is a smooth function, it changes very little within the narrow range of wavenumbers between $k_0 - \Delta k$ and $k_0 + \Delta k$.

Since the integral (4.5) is over a narrow range of wavenumbers centered on k_0, we can approximate $\phi(k)$ by a truncation of its Taylor-series expansion about k_0. The Taylor series is

$$\begin{aligned}\phi(k) &= \phi(k_0) + \phi'(k_0)(k-k_0) + \frac{1}{2}\phi''(k_0)(k-k_0)^2 + \cdots \\ &= (k_0\frac{x}{t} - \omega(k_0)) + 0 - \frac{1}{2}\omega''(k_0)(k-k_0)^2 + \cdots \end{aligned} \tag{4.6}$$

where the second term vanishes on account of (4.4). Keeping only the first two non-vanishing terms in (4.6), substituting the result into (4.5), and using the trigonometric identity (2.2), we obtain

$$\eta_R(x,t) \approx A(k_0)\cos(\theta_0)I_1(t) + A(k_0)\sin(\theta_0)I_2(t) \tag{4.7}$$

where $\theta_0 \equiv k_0 x - \omega(k_0)t$ and

$$I_1(t) = \int_{k_0-\Delta k}^{k_0+\Delta k} dk\ \cos(\frac{t}{2}\omega''(k_0)(k-k_0)^2) \tag{4.8a}$$

$$I_2(t) = \int_{k_0-\Delta k}^{k_0+\Delta k} dk\ \sin(\frac{t}{2}\omega''(k_0)(k-k_0)^2) \tag{4.8b}$$

The rest of the problem is just mathematics. To perform the integrals in (4.8), we change the integration variable from k to

$$\alpha = \sqrt{\frac{t}{2}|\omega''(k_0)|}(k-k_0) \tag{4.9}$$

Then, keeping in mind that $\omega''(k_0)$ is negative in our case, we have

$$I_1(t) = \sqrt{\frac{2}{t|\omega''(k_0)|}} \int_{-\Delta\alpha}^{+\Delta\alpha} d\alpha \ \cos(\alpha^2) \tag{4.10a}$$

$$I_2(t) = \sqrt{\frac{2}{t|\omega''(k_0)|}} \int_{-\Delta\alpha}^{+\Delta\alpha} d\alpha \ \sin(-\alpha^2) \tag{4.10b}$$

where

$$\Delta\alpha = \sqrt{\frac{t}{2}|\omega''(k_0)|}\Delta k \tag{4.11}$$

The limit $t \to \infty$ corresponds to $\Delta\alpha \to \infty$ for fixed Δk. Since

$$\lim_{\Delta\alpha\to\infty} \int_{-\Delta\alpha}^{+\Delta\alpha} d\alpha \ \cos(\alpha^2) = \int_{-\infty}^{+\infty} d\alpha \ \cos(\alpha^2) = \sqrt{\frac{\pi}{2}} \tag{4.12a}$$

$$\lim_{\Delta\alpha\to\infty} \int_{-\Delta\alpha}^{+\Delta\alpha} d\alpha \ \sin(\alpha^2) = \int_{-\infty}^{+\infty} d\alpha \ \sin(\alpha^2) = \sqrt{\frac{\pi}{2}} \tag{4.12b}$$

we obtain

$$I_1(t) = -I_2(t) = \sqrt{\frac{\pi}{t|\omega''(k_0)|}} \tag{4.13}$$

so, finally,

$$\begin{aligned} \eta_R(x,t) &\approx A(k_0)\sqrt{\frac{\pi}{t|\omega''(k_0)|}}(\cos\theta_0 - \sin\theta_0) \\ &= A(k_0)\sqrt{\frac{2\pi}{t|\omega''(k_0)|}} \cos(k_0 x - \omega(k_0)t + \frac{\pi}{4}) \end{aligned} \tag{4.14}$$

The integrals in (4.12) can be looked up.

According to (4.14), the surface elevation far from the storm is a single, slowly varying wave. According to (4.4), its wavenumber k_0 is the wavenumber of the wave that travels, *at its group velocity*, the distance x between us and the storm, in the time t since the storm. Here is further evidence that the wave energy travels at the group velocity. We see a *single* wave because we are so far from the storm. Only one particular wave has the right wavenumber to reach our location in the time since the storm.

How has the vast separation between us and the storm been built into our calculation? In two ways. First, by our assumption that $t \to \infty$ (for fixed x/t). And second, by our assumption that $A(k)$ is a smooth function. The latter assumption is equivalent to the assumption that the initial surface distribution $\eta(x,0) = F(x)$ is concentrated near $x = 0$. How do we see this? Suppose, for example, that $F(x) = F_0 e^{-\beta x^2}$ with F_0 and β positive constants. We have used this example before! The larger the constant β, the more $F(x)$ is concentrated near $x = 0$. For this choice of $F(x)$, (3.26) gives

$$A(k) = \frac{1}{\pi}\int_0^\infty dx\; F(x)\cos(kx) = \frac{F_0}{2\sqrt{\pi\beta}} e^{-k^2/4\beta} \tag{4.15}$$

where we have used the formula (3.19). We can describe this $A(k)$ as a hump whose width (in k-space) is inversely proportional to the width of $F(x)$ (in x-space). Thus, as $\beta \to \infty$ ($F(x)$ sharply concentrated at $x = 0$), $A(k)$ becomes an infinitely wide hump. That is, $A(k)$ becomes very smooth.

What is true in this particular example is true in general. The more concentrated the $F(x)$, the more spread out is its Fourier transform $A(k)$. When you study quantum mechanics, you will see that *Heisenberg's Uncertainty Principle* corresponds to this same mathematical fact. In that context, and speaking very very roughly, $F(x)$ is the probability of finding a particle at location x, and $A(k)$ is the probability that the particle has velocity k. (Never mind the wrong units—Planck's constant fixes that!) Just remember that you heard it first in a course on ocean waves.

To better admire our final answer, we re-write (4.14) in the form of a *slowly varying wavetrain*,

$$\eta_R(x,t) \approx A_{SV}(x,t)\cos\left(k(x,t)x - \omega(x,t)t + \frac{\pi}{4}\right) \tag{4.16}$$

Here,

$$A_{SV}(x,t) = A(k(x,t))\sqrt{\frac{2\pi}{t|\omega''(k(x,t))|}} \tag{4.17}$$

is the slowly varying amplitude; $k(x,t)$ is the slowly varying wavenumber; and $\omega(x,t)$ is the slowly varying frequency. By *slowly varying* we mean that these three quantities change by only a small percentage over each wavelength or period.

Let (x,t) be given. The slowly varying wavenumber at (x,t) is determined as the solution to (4.4), namely

$$\omega'(k) = x/t \tag{4.18}$$

For deep water waves, this would be

$$\text{DW} \qquad \frac{1}{2}\sqrt{\frac{g}{k}} = x/t \quad \text{so} \quad k(x,t) = \frac{gt^2}{4x^2} \tag{4.19}$$

If x and t are large, it is obvious that $k(x,t)$ changes very little when x changes by a wavelength, or when t changes by a wave period. With $k(x,t)$ thus determined, $\omega(x,t)$ is determined by the dispersion relation

$$\text{DW} \qquad \omega = \sqrt{gk} \quad \text{so} \quad \omega(x,t) = \sqrt{gk(x,t)} = \frac{gt}{2x} \tag{4.20}$$

Obviously, $\omega(x,t)$ is also a slowly varying function. Finally, the slowly varying amplitude is obtained by substituting $k(x,t)$ into (4.17).

If all these things vary slowly, what is it that changes rapidly? The answer, of course, is $\eta(x,t)$ itself; it changes by 100% in each wavelength and period.

How do we understand the form (4.17) taken by the slowly varying amplitude? What is it telling us? Imagine that you and a friend each have speed boats, and you decide to play a little game with this slowly varying wavetrain. Each of you picks a fixed value of wavenumber, and each of you decides to drive your boat at just the right speed to always observe your chosen wavenumber. If you choose the wavenumber k_1, then the location of your boat must satisfy $\omega'(k_1) = x/t$. In other words, you must drive your boat at the group velocity corresponding to k_1. If your friend chooses the value k_2, she must drive her boat at the group velocity $\omega'(k_2)$ corresponding to k_2. We suppose that $k_2 < k_1$; your friend has decided to follow a longer wavelength than yours.

Driving these boats will take some skill. You can't be fooled into following your wavenumber by keeping up with crests and troughs. If you do that, you will notice that the wavelength you observe will gradually get longer. You will have left your assigned wavenumber far behind. To keep pace with your assigned wavenumber, you must drive your boat at *half* the speed of the crests and troughs, because in deep water the group velocity is *half* the phase velocity. Up ahead, your friend must do the same for her assigned wavenumber k_2. But since $k_2 < k_1$, her boat will move faster than yours. The two boats gradually draw apart.

If energy really moves at the group velocity, then the total amount of energy between the two boats must always be the same. From the previous chapter we know that the energy per unit horizontal distance is proportional to the square of the slowly varying wave amplitude. Therefore, if energy moves at the group velocity, it must be true that

$$\int_{x_1(t)}^{x_2(t)} dx\ A_{SV}(x,t)^2 = \text{constant} \tag{4.21}$$

where $x_1(t)$ is the location of your boat, and $x_2(t)$ is the location of your friend's boat. Suppose that the difference between k_1 and k_2 is very small. Then the difference between $x_1(t)$ and $x_2(t)$ is also very small, and (4.21) becomes

$$(x_2(t) - x_1(t)){A_{SV}}^2 = (\omega'(k_2) - \omega'(k_1))t{A_{SV}}^2 = \text{constant} \tag{4.22}$$

where $\omega'(k_2) - \omega'(k_1)$ is the difference in the boat speeds. If the difference between k_1 and k_2 is very small, then

$$\omega'(k_2) - \omega'(k_1) \approx \omega''(k_1)(k_2 - k_1) \tag{4.23}$$

Thus (4.22) implies that

$${A_{SV}}^2 \propto \frac{1}{t|\omega''(k_1)|} \tag{4.24}$$

which agrees with (4.17) and provides an explanation for it. The square of the slowly varying amplitude—the energy density—is inversely proportional to the separation between the two boats, and it decreases as the boats diverge. The energy per unit distance decreases because the same amount of energy is spread over a wider area.

Our calculation shows why, if you are surfer, it is better to be far away from an intense, wave-producing storm than close to it. If the storm were just over the horizon, the waves reaching you could not yet have dispersed. You would be seeing a superposition of all the wavelengths produced by the storm. The surf would be a jumble. In the case of a distant storm, you see only the wavelength whose group velocity matches your location and time. Even if the storm covers a wide area or lasts for a significant time, you will still see a single wavetrain if the distance between you and the storm is sufficiently great.

If the situation is more complicated, our solution still has value, because Postulate #2 says that waves superpose. For example, suppose there are *two* storms, well separated from one another, but both very far away. Then the sea state at your location will be a superposition of two slowly varying wavetrains, each determined by its distance in space and time from its source. These two wavetrains will interfere, just like the two waves traveling in one direction that we considered in chapter 2. The result could be a series of wave groups—sets—like those described in chapter 2. The sets move at the average group velocity of the two waves.

If you are beginning to get the idea that group velocity is all-important, then you are absorbing one of the central ideas of this course. In dispersive waves like DW, the group velocity is the key concept, and it is much more important than the phase velocity. In nondispersive waves like SW, the group velocity and the phase velocity are the same, so there is no need to introduce the concept of group velocity. In nondispersive systems the phase velocity assumes great importance. For example, the solution (3.32) involves the phase velocity.

The two types of waves most frequently encountered in engineering and physics courses are electromagnetic waves and acoustic waves. Both of these are, to good approximation, nondispersive waves. Electromagnetic waves are exactly nondispersive in vacuum, but become slightly dispersive in material media. The phenomenon of chromatic aberration in optics is one manifestation of the slight dispersion of electromagnetic waves in glass. However, dispersive effects in electromagnetism and acoustics are usually sufficiently small that the concept of group velocity isn't invoked. This explains why you may never have heard of it, even if you have studied these waves. In water waves, group velocity becomes the key concept. There is no way to avoid it.

The mathematical techniques for dealing with these two classes of waves—dispersive and nondispersive—differ greatly. For dispersive waves like DW, the primary techniques are Fourier analysis and the superposition of waves—the very techniques you have begun to learn. Of course these methods also work—as a special case—for nondispersive waves like SW. But for nondispersive waves there are, in addition, very powerful and specialized mathematical methods that apply *only* to nondispersive waves. These specialized methods are useful for studying *shocks*, which occur commonly in nondispersive systems. (The SW analog of the shock is the *bore*, which we will discuss in chapter 10; breaking ocean waves often turn into turbulent bores.) However, the specialized methods, besides being limited to nondispersive waves, are mathematically rather advanced, and are somewhat beyond our scope.

Let's rewrite (4.21) in a slightly different form, as

$$\int_{x_1(t)}^{x_2(t)} E(x,t)\,dx = \text{constant} \tag{4.25}$$

where $E(x,t)$ is the average wave energy per unit area. The time derivative of (4.25) is

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{x_1(t)}^{x_2(t)} E(x,t)\,dx \\
&= \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t}E(x,t)\,dx + E(x_2,t)\frac{dx_2}{dt} - E(x_1,t)\frac{dx_1}{dt} \\
&= \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t}E(x,t)\,dx + E(x_2,t)c_g(x_2,t) - E(x_1,t)c_g(x_1,t)
\end{aligned} \tag{4.26}$$

because the speed boats are moving at the group velocity. The last line can be written in the equivalent form

$$0 = \int_{x_1}^{x_2} \frac{\partial}{\partial t}E(x,t)\,dx + F(x_2,t) - F(x_1,t) \tag{4.27}$$

where $F(x,t) = c_g(x,t)E(x,t)$ is the energy flux toward positive x. This can be written yet again as

$$0 = \int_{x_1}^{x_2} \left[\frac{\partial}{\partial t}E(x,t) + \frac{\partial}{\partial x}F(x,t)\right]\,dx \tag{4.28}$$

Finally, since (4.28) must hold for every value of x_1 and x_2—that is, for every pair of constant wavenumbers selected by the two speed boat drivers—it must be true that

$$\frac{\partial}{\partial t}E(x,t) + \frac{\partial}{\partial x}F(x,t) = 0 \tag{4.29}$$

everywhere. Equivalently,

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}(c_g E) = 0 \tag{4.30}$$

We have seen this equation before!

Our calculation has been for the case of one space dimension, but the oceans surface is two-dimensional. In two dimensions, our calculation corresponds to an infinitely long storm located along the y-axis. It is more realistic

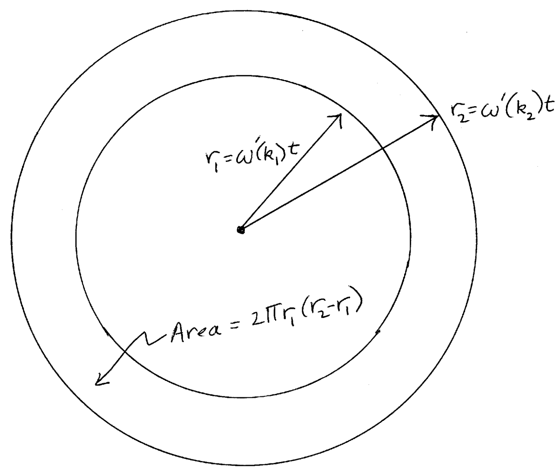

Figure 4.2: The area of the annulus increases because both r_1 and $(r_2 - r_1)$ increase.

to regard the storm as a disturbance located near the *point* $(x, y) = (0, 0)$ at time $t = 0$. In this more realistic case, the waves spread away from the storm in a pattern of concentric circles. As in the one-dimensional case, energy moves at the group velocity, but in two dimensions the slowly varying amplitude is given by

$$A_{SV}{}^2 \propto \frac{1}{t^2|\omega'(k_1)\omega''(k_1)|} \tag{4.31}$$

instead of (4.24). To understand (4.31), we must use our speed boats again. The first speed boat driver, traveling at just the right speed to always observe wavenumber k_1, must stay at radius $r_1 = \omega'(k_1)t$ measured from the center of the storm (figure 4.2). The second speed boat driver, always observing k_2, must stay at $r_2 = \omega'(k_2)t$. The energy in the annulus between r_1 and r_2 is conserved. Hence, assuming that the difference between r_1 and r_2 is small,

$$\text{constant} = 2\pi r_1(r_2 - r_1)A_{SV}{}^2 = 2\pi(\omega'(k_1)t)(|\omega''(k_1)|t\,\Delta k)A_{SV}{}^2 \tag{4.32}$$

where $\Delta k = |k_2 - k_1|$. In two dimensions, the wave amplitude decreases faster than in one dimension because the same amount of energy is spread over an *even larger* area than in one dimension. Part of the enlargement is caused by the dispersion of waves (increasing separation of the speed boats) in the radial direction of wave propagation, just as in the one-dimensional case. However, now there is an additional enlargement caused by the fact that the circles themselves get larger.

We should admit that in talking about wave energy we have gone somewhat beyond the authority of Postulates #1 and #2. We have introduced the additional assumption that the average energy density—the energy per unit area of ocean surface—is proportional to the *square* of the wave amplitude. In a chapter 2 we tried to justify this assumption with an incomplete calculation of the kinetic and potential energies, but we were mainly relying upon what you already know about energy. Postulates #1 and #2 say nothing at all about energy! When we eventually get around to justifying Postulates #1 and #2 by considering the general equations for a fluid, we will need to verify energy conservation as well.

Chapter 5

Wave measurement and prediction

In chapter 4 we considered the waves created by a highly idealized storm: a point disturbance in space and time, with a symmetrical shape that was designed to make our calculation as easy as possible. Although this type of calculation can teach us a lot about the basic physics of ocean waves, the real ocean is a much more complicated place. In this lecture we acknowledge that complexity by discussing the methods by which real ocean waves are measured and predicted.

If simple calculations could explain everything, there would be no need for measurements at all. To the contrary, oceanography is an empirical science in which the measurements always guide the theory. But measurements by themselves can only tell you what has already happened. For anyone living or working near the ocean, accurate predictions of wave height, at least several days ahead, are vitally important. These predictions rely on both measurements and theory. How are they made?

When talking about ocean waves, oceanographers often refer to the *energy spectrum* $S(\mathbf{k}, \mathbf{x}, t)$, defined as the amount of energy in the wave with wave vector $\mathbf{k} = (k, l)$ at the location $\mathbf{x} = (x, y)$ and the time t. More precisely,

$$\int_{k_1}^{k_2} dk \int_{l_1}^{l_2} dl \; S(k, l, x, y, t) \tag{5.1}$$

is the energy in all the waves with wavenumbers (k, l) in the range $k_1 < k < k_2$ and $l_1 < l < l_2$, at location $\mathbf{x}$ and time t. If we integrate $S(\mathbf{k}, \mathbf{x}, t)$ over *all* wavenumbers, we get the total average energy, per unit mass, per unit area,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\mathbf{k}\; S(\mathbf{k},\mathbf{x},t) = g\langle\eta^2\rangle \tag{5.2}$$

where $\langle\ \rangle$ denotes the average in the vicinity of $\mathbf{x}$; see (2.36). Thus $S(\mathbf{k},\mathbf{x},t)$ has the units L^5T^{-2}. Note that $\mathbf{k}$, $\mathbf{x}$, and t are *independent* variables. That means that you get to choose the time, the place, and the wave vector $\mathbf{k}$ of the waves you want to know about. The function S tells you how much energy these waves have at the place and time you have chosen. Specifying $\mathbf{k}$ is the same as specifying the direction μ of wave propagation and the frequency ω, so the spectrum could also be taken as $S(\mu,\omega,\mathbf{x},t)$.

The spectrum $S(\mathbf{k},\mathbf{x},t)$ is somewhat similar to the energy density $E(\mathbf{x},t)$ considered in chapter 4. However, $E(\mathbf{x},t)$ depended only on location and time. This is because the wave field in chapter 4 consisted of only a *single* wave vector at every location. That wave vector, $\mathbf{k}(x,y,t)$, differed from one location to another, but there was never more than one k present at any location. In defining the spectrum, we are admitting that the real world is more complicated; waves with every wavelength and direction can be present simultaneously at the same place. $S(\mathbf{k},\mathbf{x},t)$ tells us which ones have the most energy.

How is the spectrum calculated? We cannot go into details (which become rather technical), but the spectrum is calculated in much the same way as the wave amplitudes $A(k)$ and $B(k)$ in chapter 3. In fact, the one-dimensional spectrum is proportional to $A(k)^2+B(k)^2$, the sum of the squares of the wave amplitudes. The two-dimensional spectrum is proportional to $A(\mathbf{k})^2+B(\mathbf{k})^2$. To compute $A(\mathbf{k})$ and $B(\mathbf{k})$, we use a formula similar to, but subtly different from, (3.14). The subtle difference is that the integration limits can no longer be infinite, because we are calculating the spectrum near the specified location $\mathbf{x}$, and if we use values of $\eta(\mathbf{x}',t)$ with $\mathbf{x}'$ far from $\mathbf{x}$, then we are confusing the energy near $\mathbf{x}'$ with the energy near $\mathbf{x}$. The formula (3.14) gets modified to avoid this.

There are other kinds of spectra, besides $S(\mathbf{k},\mathbf{x},t)$. The spectrum $S(\kappa,\mathbf{x},t)$ measures the energy in all the waves with wavenumber magnitude $\kappa = |\mathbf{k}|$. The *frequency spectrum* $S(\omega,\mathbf{x},t)$ measures the energy in all the waves with frequency ω. Both $S(\kappa,\mathbf{x},t)$ and $S(\omega,\mathbf{x},t)$ contain less information than $S(\mathbf{k},\mathbf{x},t)$; neither $S(\kappa,\mathbf{x},t)$ nor $S(\omega,\mathbf{x},t)$ contains any information about the direction in which the waves are propagating. However, even $S(\mathbf{k},\mathbf{x},t)$

does not provide a complete description of the wave field, because it does not keep track of the *phase differences* between the waves. In summary, knowledge of $\eta(\mathbf{x}, t)$ is sufficient to calculate any of these spectra, but none of the spectra contain enough information to reconstruct $\eta(\mathbf{x}, t)$ exactly. This is the main criticism aimed at spectral analysis and at prediction methods that deal only with the spectrum.

You can apply spectral analysis to any quantity $\eta(\mathbf{x}, t)$; you need not even be talking about waves. But if $\eta(\mathbf{x}, t)$ is the sea surface elevation, and if you are dealing with linear, deep-water waves, then the wavenumber and frequency are related by the dispersion relation,

$$\text{DW} \qquad \omega = \sqrt{g\kappa} \tag{5.3}$$

and this, in turn, provides a means of relating $S(\kappa, \mathbf{x}, t)$ and $S(\omega, \mathbf{x}, t)$. Since

$$\text{DW} \qquad \int_{\kappa_1}^{\kappa_2} d\kappa \; S(\kappa, \mathbf{x}, t) = \int_{\sqrt{g\kappa_1}}^{\sqrt{g\kappa_2}} d\omega \; S(\omega, \mathbf{x}, t) \tag{5.4}$$

for every κ_1 and κ_2, it must be true that

$$\text{DW} \qquad S(\kappa, \mathbf{x}, t) = S(\omega, \mathbf{x}, t)\frac{d\omega}{d\kappa} = \frac{1}{2}\sqrt{\frac{g}{\kappa}}S(\omega, \mathbf{x}, t) \tag{5.5}$$

But how do we measure $\eta(\mathbf{x}, t)$?

In situ measurements are those in which the measuring device is located near the point of measurement. A simple *in situ* device consists of a pair of uninsulated vertical wires extending through the sea surface, with a voltage difference applied between the wires. Since seawater conducts electricity, the current flowing from one wire to the other is proportional to $\eta(\mathbf{x}, t)$. A similar device uses a single insulated wire and measures the change in capacitance of the wire. Such wire devices are mainly useful in shallow water, where the sea bottom offers a stable support. Other shallow-water wave-measuring devices include pressure sensors, which estimate the sea surface elevation from measurements of the fluid pressure near the sea bottom. In its vertical position, the Scripps research vessel FLIP provides a stable platform in deep water. However, most wave measurements in deep water are made by unattended, moored buoys. Heave-pitch-roll buoys (such as the Waverider Buoy) contain accelerometers that measure the acceleration of the water in three directions. The measured accelerations are radioed to shore, via satellite if necessary, and converted into estimates of the wave energy spectrum. Although *in situ* measurements like those from buoys are very accurate, they have a relatively small coverage. They only measure wave conditions near the device itself. Accurate wave prediction requires that we keep track of $S(\mathbf{k}, \mathbf{x}, t)$ over the entire world ocean. Only satellites can do this.

Remote sensing of the ocean surface by satellites has been possible for about 30 years. Two types of instruments measure ocean waves. *Altimeters* determine the distance between the satellite and the ocean surface by measuring the time required for a radar pulse to bounce off the sea surface and return to the satellite. The time difference between the pulses that reflect from wave crests and those that reflect from wave troughs provides a rough measure of total wave energy. This measure is usually converted to *significant wave height* H_s, defined as the average height of the 33% largest waves (or something similar). This definition is chosen to make H_s agree with the wave height (measured crest to trough) estimated by a typical human observer. The JASON-1 satellite, launched in December 2001 by the U.S. and France, carries a dual altimeter operating at frequencies of 5.3 GHz and 13.6 GHz, corresponding to wavelengths of 6 cm and 2 cm, respectively. JASON-1 transmits a one-meter-long radar pulse, which measures sea surface elevation to an accuracy of 2-3 cm, and can determine H_s to an accuracy of 5% or 25 cm (whichever is greater) as compared with buoy measurements. Each measurement represents the average of H_s along about 10 km of the satellite's track. Considering that the average H_s exceeds 5 m in stormy latitudes, this is good accuracy. Unfortunately, H_s is a single number, which tells us nothing about how the wave energy is distributed among the various wavelengths and directions.

The second type of satellite instrument used for measuring ocean waves is the *synthetic aperture radar* or SAR, which transmits a radar wave of 3 to 25 cm wavelength at an oblique angle to the ocean surface. The satellite measures the backscattered wave. The wave frequency is chosen for its ability to penetrate the atmosphere. At frequencies lower than 1 GHz (wavelengths longer than 25 cm) there is significant contamination from the cosmic microwave background and from microwave radiation emanating from the center of the Milky Way galaxy. At frequencies larger than 10 GHz (wavelengths shorter than 3 cm), atmospheric absorption becomes a problem. Since the radar waves are transmitted at an angle between 20 and 50 degrees to the ocean surface, the scattering is *Bragg scattering*, meaning that the radar wave interacts with an ocean wave having the same 3-25 cm wavelength as the radar. Ocean waves with these wavelengths are in the gravity-capillary range. Such waves are always present, but they have much less energy than the longer ocean waves, which are therefore of greater interest. However, it is possible to infer the size and direction of the longer waves by observing the *modulation* they impose on the capillary waves. For example, the largest-amplitude capillary waves tend to occur on the forward side of swell crests.

SAR measurements of ocean swell demand a field of view that is smaller than the wavelength of the swell. Such a narrow beam would normally require an impractically large antenna on the satellite itself. For example, an *ordinary* radar using a satellite antenna with typical 10 m width has a field of view of about 10 km. This is much too large; the longest swells have wavelengths of only a few hundred meters. To narrow the field of view, SAR makes use of the satellite's rapid motion along its track. Very roughly speaking, measurements from slightly different locations along the track are treated like the signal detected from the different elements of a large antenna. This reduces the SAR's field of view to about 25 m. Because of the directional nature of the SAR signal, SARs provide information about the direction of the waves, and can even be used to estimate $S(\mathbf{k}, \mathbf{x}, t)$.

Ordinary, non-SAR radars offer an excellent means of measuring the wind speed above the ocean. Because capillary-wave amplitudes are strongly correlated with the local wind, these radars, usually called *scatterometers*, provide accurate estimates of the wind velocity within a 10 km patch. By looking at the same patch of ocean from several directions, scatterometers can measure both wind speed and direction to an accuracy of about 10%.

Of course, even the best measurements cannot tell us what the waves will be like three days from now. Oceanographers predict the future wave spectrum $S(\mathbf{k}, \mathbf{x}, t)$ by solving the spectral evolution equation

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(c_{gx} S) + \frac{\partial}{\partial y}(c_{gy} S) = Q_{wind} + Q_{diss} + Q_{transfer} \tag{5.6}$$

over the whole world ocean, using powerful computers. The left-hand side of (5.6), in which

$$\mathbf{c}_g = (c_{gx}, c_{gy}) = (\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}) \tag{5.7}$$

is the group velocity, contains the only terms that would be present if the waves were linear, unforced, and undissipated—the only case considered in the previous chapters. The right-hand side of (5.6) contains the sources and sinks of wave energy. For deep water waves,

$$\text{DW} \qquad \omega = \sqrt{g\kappa} = \sqrt{g(k^2 + l^2)^{1/2}} \tag{5.8}$$

and hence the group velocity is given by

$$\text{DW} \qquad \mathbf{c}_g = \frac{1}{2}\sqrt{\frac{g}{\kappa}}(\frac{k}{\kappa}, \frac{l}{\kappa}) \tag{5.9}$$

where $(k/\kappa, l/\kappa)$ is the unit vector in the direction of wave propagation.

The left-hand side of (5.6) resembles (4.30) and has a similar derivation. However, (5.6) is not the most general prediction equation that could be used. It neglects wave refraction and other effects of ocean currents. If ocean currents are present, then the waves can exchange energy with the currents, and the whole calculation is best reformulated in terms of the spectrum of a quantity called *wave action*. The details of this are beyond our scope. For the purposes of our discussion, (5.6) is accurate enough. In fact, our main purpose here is a brief, qualitative discussion of the three source/sink terms on the right-hand side of (5.6).

The first of these terms represents the energy put into the waves by the wind. The strongest winds occur in the two belts of mid-latitude cyclonic storms at about 50 degrees latitude in both hemispheres. The source of the wind-energy in these storms is the potential energy created by the sun's unequal heating of the Earth's surface. These storms are strongest in the winter hemisphere. Thus the Southern Ocean (the area around Antarctica) is the primary source of San Diego's summer swell. Our winter swell originates in the North Pacific, and is stronger because we are closer to the source. Hurricanes, whose wind-energy comes from the latent heat created by the evaporation of tropical waters, also generate large swell, but hurricanes are relatively small contributors to the global wave budget.

The exact mechanism by which wind creates ocean waves remained a mystery until the 1950's. Then, in 1957, UCSD Professor John Miles proposed a theory that is still considered to be the most successful. A complete discussion of Miles's theory requires a rather deep understanding of fluid mechanics. Put simply, Miles proposed that the wind flowing over an ocean wave generates a corresponding wave in the air above the ocean. These two waves conspire to draw energy out of the wind. By Miles's calculation,

$$Q_{wind} = \frac{\rho_a}{\rho_w}\beta\omega(\frac{U_r}{c}\cos\theta)^2 S(\mathbf{k}, \mathbf{x}, t) \tag{5.10}$$

where ρ_a is the mass density of the air in grams per cubic centimeter; ρ_w is the mass density of the water; U_r is the average wind speed at a reference level of 10 meters above the ocean surface; and θ is the angle between the wind and the direction of wave propagation at the phase velocity c. The quantity β is an order-one, dimensionless function of U_r/c that decreases rapidly as U_r falls below c. This makes sense; the wind cannot do work on the wave if the wave is traveling faster than the wind. Subsequent refinements of the theory have proposed various forms for β.

Since the wind forcing (5.10) is *proportional* to $S(\mathbf{k}, \mathbf{x}, t)$, it leads to exponential growth of the wave spectrum. On the other hand, if $S(\mathbf{k}, \mathbf{x}, t) = 0$, then Q_{wind} vanishes as well. Thus the Miles theory predicts wave growth only if waves are already present. It cannot explain how waves first develop in a calm sea. An alternative theory proposed by Owen Phillips fills the gap. Phillips proposed that ocean waves first arise in response to random pressure fluctuations in the atmosphere just above the sea surface. Because these fluctuations are turbulent, they occur at every wavelength and frequency, including those wavelengths and frequencies that match the dispersion relation for ocean waves. This causes the waves to grow, but at a rate that is *independent* of S, and therefore linear—not exponential—in the time t. This relatively slow growth is enough to get things started. Wave prediction models typically use a form of Q_{wind} that incorporates both the Miles mechanism and the Phillips mechanism.

Now, what about the next term, Q_{diss}? This represents the loss of wave energy to dissipation. In the open ocean, the dominant mechanism of dissipation is wave-breaking, i.e. white-capping. This mechanism is very poorly understood, but a combination of crude physical reasoning and empiricism leads to parameterizations like

$$Q_{diss} = -C_{diss}\omega\kappa^8 S(\mathbf{k}, \mathbf{x}, t)^3 \tag{5.11}$$

where C_{diss} is a constant. Note that the size of (5.11) decreases rapidly as the wavelength increases. Thus white-capping affects the shortest waves the most.

The last term in (5.6) is neither a net source nor a net sink of wave energy. Instead, $Q_{transfer}$ represents the transfer of wave energy between waves with different wavelengths; $Q_{transfer}$ conserves energy overall. The energy transfer represented by $Q_{transfer}$ arises from the *nonlinear* terms in the equations of fluid motion, and it does not occur in the approximate, linear dynamics

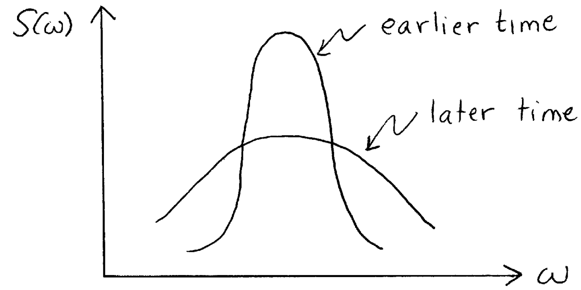

Figure 5.1: Spreading of energy by nonlinear transfer, $Q_{transfer}$.

governed by Postulates #1 and #2. (All of this will become a little clearer in chapter 8.) $Q_{transfer}$ is better understood than Q_{diss}, but it takes a much more complicated form. The complexity arises from the fact that wave energy transfer takes place via *resonant quartets* of ocean waves. Because of this, the exact expression for $Q_{transfer}$ involves multiple integrals over wavenumber space. The evaluation of $Q_{transfer}$ is by far the most time-consuming step in the solution of the wave prediction equation (5.6). Although the exact form of $Q_{transfer}$ is quite complicated, its net affect is easy to describe. $Q_{transfer}$ removes energy from those $\mathbf{k}$ that have lots of energy, and gives the energy to other $\mathbf{k}$ that don't have as much. That is, $Q_{transfer}$ causes the spectrum to spread out (figure 5.1).

We can understand how all this fits together by considering a special case. Suppose that a storm covers the whole ocean, and that its intensity is everywhere the same. Then $S(\mathbf{k}, \mathbf{x}, t) = S(\mathbf{k}, t)$ is everywhere the same and is governed by

$$\frac{\partial}{\partial t} S(\mathbf{k}, t) = Q_{wind} + Q_{diss} + Q_{transfer} \tag{5.12}$$

Compare this to (5.6). As the storm continues, the waves build up, but Q_{diss} increases faster than Q_{wind}. Eventually, an equilibrium is reached in which $S(\mathbf{k}, t) = S_{eq}(\mathbf{k})$ and (5.12) reduces to

$$0 = Q_{wind} + Q_{diss} + Q_{transfer} \tag{5.13}$$

This equilibrium spectrum depends only on the wind strength. What does it look like?

According to one idea, the equilibrium spectrum represents a saturated state in which any further wave growth is rapidly cancelled by wave breaking.

Wave breaking occurs when the acceleration of the fluid particles approaches the gravitational acceleration g. Plausibly, then, the equilibrium frequency spectrum $S_{eq}(\omega)$, which has units L^3T^{-1}, should depend only on g (with units LT^{-2}) and ω (with units T^{-1}). The only dimensionally consistent possibility is

$$S_{eq}(\omega) = Cg^3\omega^{-5} \tag{5.14}$$

where C is a dimensionless constant. By (5.5), this corresponds to the equilibrium wavenumber spectrum,

$$S_{eq}(\kappa) = \frac{1}{2}Cg\kappa^{-3} \tag{5.15}$$

However, observations seem to favor a spectrum proportional to ω^{-4} over that predicted by (5.14). Of course the spectrum does not extend to indefinitely low ω and κ, because waves whose phase speed exceeds the wind speed would tend to be retarded rather than pushed by the wind. Sustained winds of more than 50 m/sec (about 100 miles per hour) are very unusual. In deep water, a phase speed of 50 m/sec corresponds to a wavelength of 1500 m and a period of 30 seconds. Wave spectra contain very little energy at longer periods and wavelengths than these.

Beginning from a calm initial state, how does the saturation spectrum develop? Observations show that the shortest waves grow fastest, quickly reaching their saturation level. As the short waves saturate, they begin to transfer some of their energy to longer waves. The peak in the spectrum moves left, toward lower frequency. The spectral peak is higher than the spectrum at that *same* frequency at a later time, suggesting that the longest waves present at any particular time have an advantage over other wavelengths in capturing the energy of the wind. See figure 5.2.

In the more realistic case of a storm of finite size and duration, we must solve the full equation (5.6) to determine the wave field outside the storm. How is this done? Since computers can't solve differential equations, (5.6) is converted to a *difference equation* as follows. Imagine that the world ocean is covered by a square mesh in which the meshpoints are separated by a distance d. Let $S_{ij}^n(\mathbf{k})$ be the spectrum at the meshpoint $(x, y) = (id, jd)$ at time $t = n\tau$, where τ is the time step. For the sake of simplicity, we specialize to the one-dimensional case. A difference equation corresponding

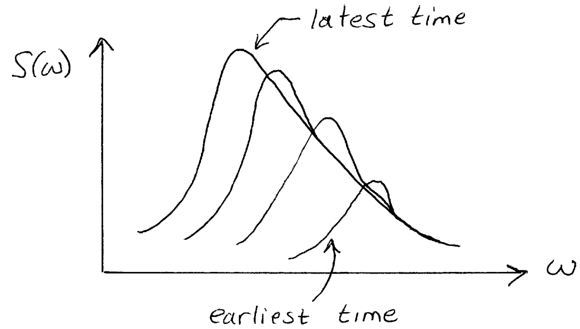

Figure 5.2: Growth of the energy spectrum with time.

to the one-dimensional form of (5.6) is

$$\frac{(S_i^{n+1}(\mathbf{k}) - S_i^n(\mathbf{k}))}{\tau} + \frac{(c_g S)_{i+1}^n - (c_g S)_{i-1}^n}{2d} = (Q_{wind} + Q_{diss} + Q_{transfer})_i^n \tag{5.16}$$

If the spectrum is known at some initial time, then (5.16) may be used to find it at the next time step. This process can be iterated as far into the future as needed. The WAM model (acronym for WAve Model) is the best known numerical model of this general type. It uses a mesh spacing d of about 200 km, and a time step τ of about 20 minutes.

To compute Q_{wind} we need to know the forecast winds, and these must be computed from a global weather model similar in principle to (5.16). The weather model uses the observed weather as its starting condition. Similarly, (5.16) would seem to require the observed wave spectrum as its starting condition. Fortunately, however, the computed wave spectrum adjusts so rapidly to the right-hand side of (5.16) that the initial condition for (5.16) is much less important than the initial condition for the weather model. The solution to (5.16) is dominated by the wind, and errors in the wind forecast seem to be the biggest source of error in wave predictions.

How well do the global wave forecast models work? Their accuracy compared to open-ocean buoy observations is very impressive. But the global models only predict the waves in deep water. They treat the coastlines as perfectly absorbing boundaries. This is realistic; the amount of wave energy reflected from natural shorelines is quite small. However, the precise way in which waves shoal and break is of great importance, and it depends on details of the local coastal bathymetry that could not possibly be considered in a global model.

Figure 5.3 is from the website of the SIO Coastal Data Information Program (http://cdip.ucsd.edu). It shows the wave height on 7 December 2015 in the Southern California Bight, predicted by a model that incorporates local bathymetry. This particular model uses the wave spectrum measured at the Harvest Buoy off Point Conception—rather than the output of a global prediction model—to prescribe the incoming swell. The model includes the effects of wave diffraction and refraction by islands and bathymetry, but it ignores the processes represented by the right-hand side of (5.6). The directional spectrum at the lower left of the figure shows 17-second waves with significant wave height 15.5 ft arriving from the northwest. These waves were generated by a storm in the North Pacific. The offshore islands throw wave shadows that are gradually filled in by refraction. The waves reaching San Diego pass between Catalina Island and San Clemente Island.

As waves begin to feel the ocean bottom very near the shore, their paths are bent and focused by the variations in the mean ocean depth $H(x, y)$. Once again, accurate predictions require elaborate computer models. However, relatively simple theoretical tools can explain much of what is seen. The next chapter introduces those tools.

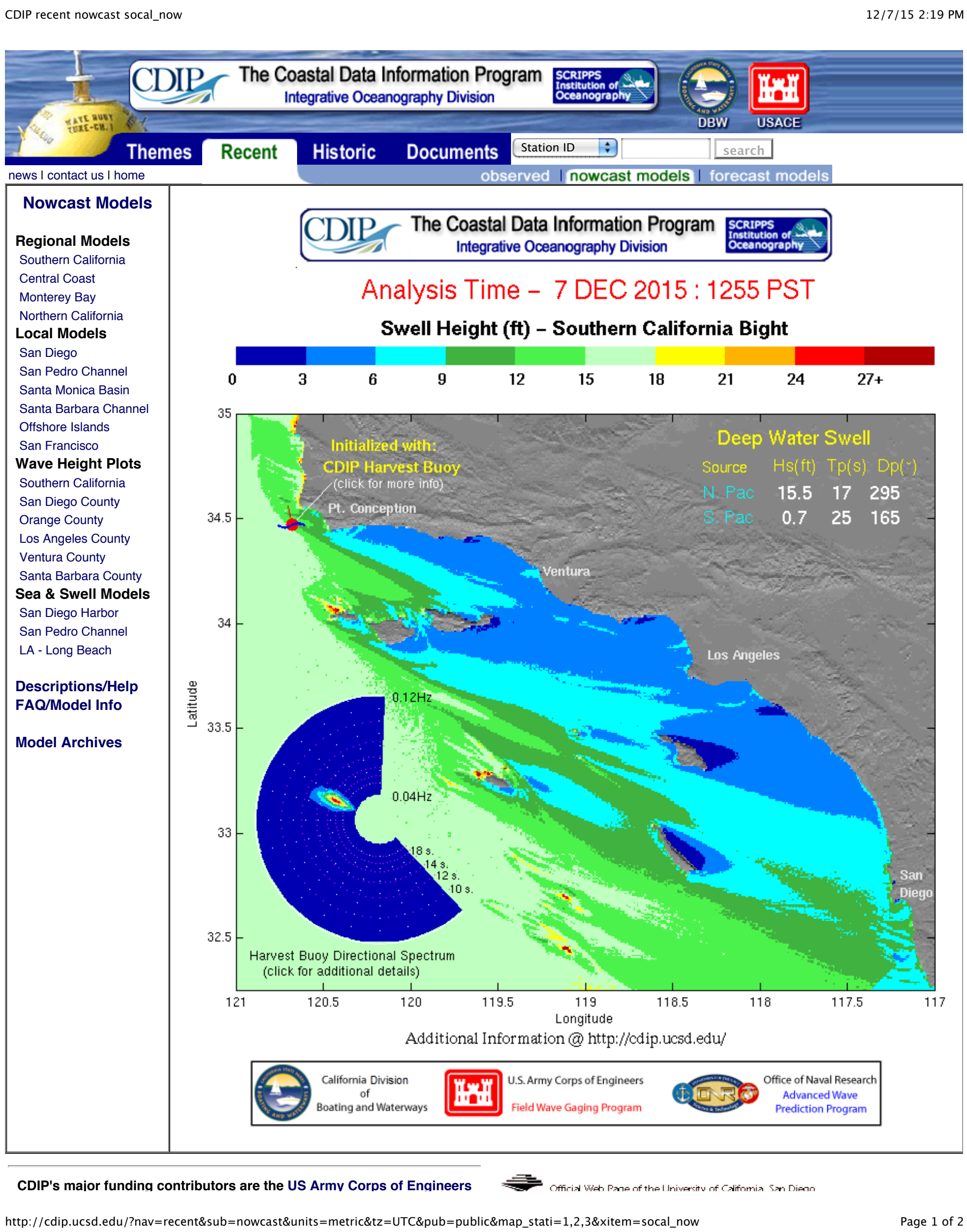

Figure 5.3: Wave height and directional spectrum in the Southern California Bight on December 7, 2015.

Chapter 6

Shoaling waves

In chapter 1, we introduced the idea of a basic wave,

$$\eta = A_0 \cos(k_0 x - \omega_0 t) \tag{6.1}$$

in which the amplitude, wavenumber, and frequency are constants. (Here, the zero subscripts serve to emphasize that the quantity is a constant.) In chapter 2, we saw how two basic waves could be combined to yield a wave

$$\eta = A_{SV}(x,t) \cos(k_0 x - \omega_0 t) \tag{6.2}$$

in which the wavenumber and frequency are constants but the amplitude varies slowly. We used the subscript SV to distinguish the slowly varying amplitude from the constant amplitudes of the basic waves. In chapter 3, we saw how an infinite number of basic waves could be added together to give a wavetrain,

$$\eta = A_{SV}(x,t) \cos(k(x,t)x - \omega(x,t)t) \tag{6.3}$$

in which the amplitude, wavenumber and frequency all vary slowly. In this chapter we explore this idea further by considering a wavetrain of the general form

$$\eta = A(x,t) \cos\theta(x,t) \tag{6.4}$$

in which the wavenumber is defined as

$$k(x,t) = \frac{\partial\theta}{\partial x} \tag{6.5}$$

and the frequency is defined as

$$\omega(x,t) = -\frac{\partial\theta}{\partial t} \tag{6.6}$$

In the case of the basic wave, $\theta(x,t) = k_0 x - \omega_0 t$, so that the wavenumber and frequency are constants, but in the general case they are not.

We require that (6.4) describe a slowly varying wave. By this we mean that $A(x,t)$, $k(x,t)$, and $\omega(x,t)$ change very little over a wavelength $2\pi/k(x,t)$ or wave period $2\pi/\omega(x,t)$. Thus, for example, we want

$$\frac{\partial k}{\partial x}\frac{2\pi}{k} \ll k \tag{6.7}$$

Written solely in terms of $\theta(x,t)$, this is

$$\theta_{xx} \ll (\theta_x)^2 \tag{6.8}$$

so another way of saying this is that the second derivatives of $\theta(x,t)$ must be much smaller than its first derivatives. Note that in (6.4) we have dropped the subscript SV from the amplitude. It is no longer needed to prevent confusion with the basic waves, because we will no longer be talking about basic waves. Of course basic waves are still present underneath—an infinite number of them, no less—but we will not refer to them explicitly.

Besides (6.8) and its analogues, we place one further restriction on $\theta(x,t)$, namely, that it obey the same dispersion relation as the corresponding basic wave. By Postulate #1, basic waves obey the dispersion relation

$$\omega_0 = \sqrt{gk_0 \tanh(k_0 H_0)} \tag{6.9}$$

We assume that the slowly varying wave obeys the *same* dispersion relation with the constant things replaced by the corresponding slowly-varying things:

$$\omega(x,t) = \sqrt{gk(x,t)\tanh(k(x,t)H(x,t))} \tag{6.10}$$

(To be completely general we have even assumed that the ocean depth depends on time.) The approximation (6.10) is reasonable, because the slowly varying wave locally resembles a basic wave. Written in terms of $\theta(x,t)$, (6.10) takes the form

$$-\theta_t = \sqrt{g\theta_x \tanh(\theta_x H)} \tag{6.11}$$

As we shall see, the differential equation (6.11) determines the phase $\theta(x,t)$ of the slowly varying wave. To determine the amplitude, we need an equation for $A(x,t)$. For this we use the previously derived result

$$E(x,t) = \frac{1}{2} g A^2(x,t) \tag{6.12}$$

and the fact that

$$\frac{\partial}{\partial t}E(x,t) + \frac{\partial}{\partial x}\left(c_g(x,t)E(x,t)\right) = 0 \tag{6.13}$$

where

$$c_g = \left.\frac{\partial \omega}{\partial k}\right|_H \tag{6.14}$$

is the slowly varying group velocity. Using the dispersion relation (6.10), we find that

$$c_g = \frac{1}{2}\sqrt{\frac{g}{k\tanh(kH)}}\left(\tanh(kH) + kH\mathrm{sech}^2(kH)\right) \tag{6.15}$$

where $k = k(x,t)$ and $H = H(x,t)$. Of course this expression simplifies considerably in the deep-water,

$$\text{DW} \qquad c_g(x,t) = \frac{1}{2}\sqrt{\frac{g}{k(x,t)}} \tag{6.15a}$$

and shallow-water limits,

$$\text{SW} \qquad c_g(x,t) = \sqrt{gH(x,t)} \tag{6.15b}$$

We determine $\theta(x,t)$ and $A(x,t)$ by solving (6.11) and (6.13) with the appropriate boundary condition. The boundary condition we have in mind is a swell wave approaching La Jolla Shores. The one-dimensional case we are examining is appropriate if the swell wave is normally incident, and if the ocean depth has no long-shore (y) dependence. These conditions are quite unrealistic; we must eventually generalize our equations to two space dimensions. However, certain pedagogical points are clearest in the one-dimensional case, so we stick to it for a bit longer.

To emphasize the general nature of what we are doing, we write the dispersion relation in the general form

$$\omega_0 = \Omega(k_0, H_0) \tag{6.16}$$

Then (6.10) becomes

$$\omega(x,t) = \Omega(k(x,t), H(x,t)) \tag{6.17}$$

and (6.11) becomes

$$-\frac{\partial\theta}{\partial t} = \Omega\left(\frac{\partial\theta}{\partial x}, H(x,t)\right) \tag{6.18}$$

If we are dealing with water waves, then $\Omega(s,H) = \sqrt{gs\tanh(sH)}$. However, by allowing Ω to be an arbitrary function, we obtain results that hold for any type of wave. In the general case, H represents an arbitrary 'medium parameter'.

We could solve (6.18) by attacking it directly, but it is better to make use of the definitions (6.5) and (6.6). By (6.5)

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial t}\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\theta}{\partial t} = -\frac{\partial\omega}{\partial x} \tag{6.19}$$

and by (6.17) this is

$$\frac{\partial k}{\partial t} = -\left.\frac{\partial\Omega}{\partial k}\right|_H \frac{\partial k}{\partial x} - \left.\frac{\partial\Omega}{\partial H}\right|_k \frac{\partial H}{\partial x} \tag{6.20}$$

Finally then,

$$\left(\frac{\partial}{\partial t} + c_g\frac{\partial}{\partial x}\right) k(x,t) = -\left.\frac{\partial\Omega}{\partial H}\right|_k \frac{\partial H}{\partial x} \tag{6.21}$$

The left-hand side of (6.21) is the time derivative of k measured by an observer traveling at the group velocity c_g. To see this, consider the wavenumber

$$k_{obs}(t) = k(x_{obs}(t), t) \tag{6.22}$$

measured by the moving observer at arbitrary location $x_{obs}(t)$. By the chain rule, its time derivative is

$$\frac{d}{dt}k_{obs}(t) = \frac{\partial k}{\partial x}\frac{dx_{obs}(t)}{dt} + \frac{\partial k}{\partial t} = \frac{\partial k}{\partial t} + u_{obs}\frac{\partial k}{\partial x} \tag{6.23}$$

where $u_{obs}(t)$ is the velocity of the observer (*aka* speed boat driver). According to (6.21), in deep water, where $\partial\Omega/\partial H = 0$, an observer moving at the group velocity observes no change in the wavenumber k. This agrees with our calculation in chapter 4. In shallow water, the same observer sees the change in k given by the right-hand side of (6.21). This change in k is called *refraction*. The corresponding change in ω is

$$\frac{\partial\omega}{\partial t} = \left.\frac{\partial\Omega}{\partial k}\right|_H \frac{\partial k}{\partial t} + \left.\frac{\partial\Omega}{\partial H}\right|_k \frac{\partial H}{\partial t} = -\left.\frac{\partial\Omega}{\partial k}\right|_H \frac{\partial\omega}{\partial x} + \left.\frac{\partial\Omega}{\partial H}\right|_k \frac{\partial H}{\partial t} \tag{6.24}$$

where we have used (6.19). Hence,

$$\left(\frac{\partial}{\partial t} + c_g \frac{\partial}{\partial x}\right) \omega(x,t) = \left.\frac{\partial \Omega}{\partial H}\right|_k \frac{\partial H}{\partial t} \tag{6.25}$$

According to (6.25), an observer moving at the group velocity sees a change in ω only if the medium H varies with time. The two fundamental equations are (6.21) and (6.25). They describe how the slowly varying wavenumber and frequency change as a function of the location and time. They cover the general case in which the medium $H(x,t)$ varies with both location and time. However, in the specific problem we are considering, the ocean depth $H(x)$ does not depend on time.

We shall assume that the waves are steady. That is, we assume that the wave conditions do not change at a fixed location. Under this assumption,

$$\frac{\partial k}{\partial t} = \frac{\partial \omega}{\partial t} = \frac{\partial E}{\partial t} = 0 \tag{6.26}$$

The conditions (6.26) will be satisfied if the incoming swell has a constant k in deep water. In this case of steady waves, the fundamental equations (6.21) and (6.25) become

$$c_g \frac{\partial}{\partial x} k(x) = -\left.\frac{\partial \Omega}{\partial H}\right|_k \frac{\partial H}{\partial x} \tag{6.27}$$

and

$$c_g \frac{\partial}{\partial x} \omega(x) = 0 \tag{6.28}$$

It follows from (6.28) that ω is a constant equal to the frequency ω_0 of the incoming swell. You may recall that we used this fact in chapter 1. To get the wavenumber $k(x)$ we could solve (6.27) for the given beach profile $H(x)$. However, it is much easier to use the dispersion relation (6.17) in the form

$$\omega_0 = \Omega(k(x), H(x)) \tag{6.29}$$

That is, it is much easier to determine $k(x)$ from

$$\omega_0 = \sqrt{g k(x) \tanh(k(x) H(x))} \tag{6.30}$$

To find the wave energy density $E(x)$, we integrate the steady form of (6.13), namely

$$\frac{\partial}{\partial x}\left(c_g(x) E(x)\right) = 0 \tag{6.31}$$

to find that

$$c_g(x)E(x) = c_{g0}E_0 \tag{6.32}$$

is a constant equal to the energy flux of the incoming swell.

We have seen (6.30) before. We derived it as (1.21) in the first chapter, with a lot less fuss. Similarly, we derived (6.32) as (2.44) in chapter 2. Why now all the elaborate discussion? The answer is that the two-dimensional shoaling problem is much harder than the one-dimensional problem, and it is the one we must solve. Real beaches have a topography $H(x, y)$ that depends both on x and on the long-shore coordinate y. Similarly, real swell is never exactly normal to the coastline. Although we may often regard the swell in deep water as an incoming basic wave, its wavevector $\mathbf{k}$ always has a long-shore component.

To discuss the two-dimensional problem, we must generalize our whole discussion, starting with

$$\eta = A(x, y, t)\cos\theta(x, y, t) \tag{6.33}$$

In two dimensions, the wavevector is defined as

$$\mathbf{k} = (k, l) = (\theta_x, \theta_y) = \nabla\theta \tag{6.34}$$

The frequency is defined, as before, by (6.6). Taking the time-derivative of (6.34), we obtain the two-dimensional generalization of (6.19),

$$\frac{\partial \mathbf{k}}{\partial t} = -\nabla\omega \tag{6.35}$$

By the slowly varying dispersion relationship

$$\omega = \Omega(k, l, H) \tag{6.36}$$

we have

$$\begin{aligned}
\nabla\omega &= \left(\frac{\partial\Omega}{\partial k}\frac{\partial k}{\partial x} + \frac{\partial\Omega}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial\Omega}{\partial H}\frac{\partial H}{\partial x}, \frac{\partial\Omega}{\partial k}\frac{\partial k}{\partial y} + \frac{\partial\Omega}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial\Omega}{\partial H}\frac{\partial H}{\partial y}\right) \\
&= \left(\frac{\partial\Omega}{\partial k}\frac{\partial k}{\partial x} + \frac{\partial\Omega}{\partial l}\frac{\partial k}{\partial y} + \frac{\partial\Omega}{\partial H}\frac{\partial H}{\partial x}, \frac{\partial\Omega}{\partial k}\frac{\partial l}{\partial x} + \frac{\partial\Omega}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial\Omega}{\partial H}\frac{\partial H}{\partial y}\right) \\
&= \left(\mathbf{c}_g\cdot\nabla k + \frac{\partial\Omega}{\partial H}\frac{\partial H}{\partial x}, \mathbf{c}_g\cdot\nabla l + \frac{\partial\Omega}{\partial H}\frac{\partial H}{\partial y}\right)
\end{aligned} \tag{6.37}$$

where we have used the fact that

$$\frac{\partial}{\partial x} l = \frac{\partial}{\partial x}\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial y} k \tag{6.38}$$

Thus the two components of (6.35) are

$$\left(\frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla\right) k = -\frac{\partial \Omega}{\partial H}\frac{\partial H}{\partial x} \tag{6.39}$$

and

$$\left(\frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla\right) l = -\frac{\partial \Omega}{\partial H}\frac{\partial H}{\partial y} \tag{6.40}$$

These are the general equations for refraction in two dimensions. To get the two-dimensional generalization of (6.25), we take the time derivative of (6.36),

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= \frac{\partial \Omega}{\partial k}\frac{\partial k}{\partial t} + \frac{\partial \Omega}{\partial l}\frac{\partial l}{\partial t} + \frac{\partial \Omega}{\partial H}\frac{\partial H}{\partial t} \\ &= -\frac{\partial \Omega}{\partial k}\frac{\partial \omega}{\partial x} - \frac{\partial \Omega}{\partial l}\frac{\partial \omega}{\partial y} + \frac{\partial \Omega}{\partial H}\frac{\partial H}{\partial t}\end{aligned} \tag{6.41}$$

Thus,

$$\left(\frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla\right) \omega = +\frac{\partial \Omega}{\partial H}\frac{\partial H}{\partial t} \tag{6.42}$$

Finally, we have the two-dimensional generalization of (6.13), namely

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{c}_g E) = 0 \tag{6.43}$$

Equations (6.39), (6.40), (6.42) and (6.43) are the fundamental equations for a slowly varying wavetrain in two dimensions. They must be solved for k, l, ω and E.

The left-hand sides of (6.39), (6.40) and (6.42) are the rates of change measured by an observer moving at the group velocity—in other words, an observer whose coordinates obey

$$\frac{dx}{dt} = \left.\frac{\partial \Omega}{\partial k}\right|_{H,l}, \qquad \frac{dy}{dt} = \left.\frac{\partial \Omega}{\partial l}\right|_{H,k} \tag{6.44}$$

Thus we may write (6.39), (6.40) and (6.42) as the *combination* of (6.44) and

$$\frac{dk}{dt} = -\left.\frac{\partial \Omega}{\partial H}\right|_{k,l} \frac{\partial H}{\partial x}, \qquad \frac{dl}{dt} = -\left.\frac{\partial \Omega}{\partial H}\right|_{k,l} \frac{\partial H}{\partial y}, \qquad \frac{d\omega}{dt} = \left.\frac{\partial \Omega}{\partial H}\right|_{k,l} \frac{\partial H}{\partial t} \tag{6.45}$$

The combination (6.44, 6.45) are called the *ray equations*. To understand them, it is best to think about speed boats again. Equations (6.44) tell us that the speed boat must be driven at the local group velocity—the group velocity obtained by substituting the local values of $k(x, y, t)$, $l(x, y, t)$ and $H(x, y, t)$ into the general expression for group velocity. We take $H(x, y, t)$ as given. (Of course, in the application we have in mind $H(x, y)$ is the *time-independent* average ocean depth.) The required values of $k(x, y, t)$ and $l(x, y, t)$ are obtained by solving (6.45), in which the right-hand sides represent the effects of refraction.

How exactly does this work? The speed boat driver starts in deep water, where the values of k and l are the (approximately) constant values k_0 and l_0 of the incoming swell. She drives her boat a short distance at the group velocity corresponding to these values. As she drives, she uses (6.45) to calculate the changes that are occurring in k and l. In deep water, no changes occur, because $\partial\Omega/\partial H = 0$ in deep water. But as the boat enters shallow water, refractive changes in k and l begin to occur. Therefore, the speed boat operator must frequently re-compute her group velocity based on the updated values of k and l. Each such re-computation demands a slight change in course of the boat. The path followed by the boat is called a *ray*. Refraction causes the rays to bend.

We can think of this process in a slightly different way, in which the boat represents a point in the four-dimensional space with coordinates (x, y, k, l). The boat moves at the four-dimensional velocity given by (6.44) and the first two equations of (6.45). In many books, these four equations are written in the form

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \qquad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i} \tag{6.46}$$

where $i = 1, 2$. This form resembles what are called *Hamilton's equations* in mechanics. The analogy between Hamilton's equations and the ray equations was a great inspiration to Shrödinger. However, the form (6.46) is dangerously ambiguous, and we will always prefer the more explicit forms (6.44) and (6.45).

The third equation of (6.45) is, in a sense, superfluous, because if we know $k(x, y, t)$ and $l(x, y, t)$, we can always compute $\omega(x, y, t)$ from the dispersion relationship. However, in the case of interest to us, $\partial H/\partial t = 0$, and this third equation then tells us that $d\omega/dt = 0$; the frequency keeps the value ω_0 of the incoming swell.

If $\partial H/\partial t = 0$, and if the amplitude, wavenumbers and frequency of the incoming swell do not vary in time, then the wave field itself will be steady. That is, the wavenumbers $k(x,y)$ and $l(x,y)$, and the wave energy $E(x,y)$ will not vary locally in time. The speed boat driver still observes a non-vanishing dk/dt and dl/dt, but only because she herself is moving though a wave field that varies with location. This is the situation we have in mind. Of course, the incoming swell *isn't* constant. It changes on a timescale of hours. But this timescale is long compared to the time required for waves to propagate from deep water to the beach. It is this separation of timescales that justifies our approximation of a constant incoming swell.

Suppose that $H = H(x)$, but that $l_0 \neq 0$; the bottom topography has no long-shore variation, but the swell is approaching the coast at an angle. In this case (6.45) tell us that *both* l and ω remain constant at their incoming values. In this case, the easiest way to find $k(x)$ is to solve the dispersion relation in the form

$$\omega_0 = \Omega\left(k(x), l_0, H(x)\right) \tag{6.47}$$

This is a problem we discussed briefly in chapter 1. The dispersion relation for water waves dictates that $k(x)$ increases as $H(x)$ decreases. In the shallow-water limit, we have

$$\text{SW} \qquad \omega_0 = \sqrt{gH(x)\left(k^2(x) + l_0^2\right)} \tag{6.48}$$

For water waves in any depth, the group velocity points in the same direction as the wavevector $\mathbf{k} = (k, l)$. (Show this!) Thus each ray satisfies the equation

$$\frac{dy}{dx} = \frac{c_{gy}}{c_{gx}} = \frac{l}{k} = \frac{l_0}{k(x)} \tag{6.49}$$

We can solve for the rays by solving (6.49) with $k(x)$ given by (6.47). However, without even touching a pencil, we can see that dy/dx must decrease as $k(x)$ increases. The rays turn toward the beach as they approach it. The wavecrests, which are perpendicular to the rays, become more nearly parallel to the beach.

Now let γ be the angle between the rays and the perpendicular to the shoreline (figure 6.1). Then

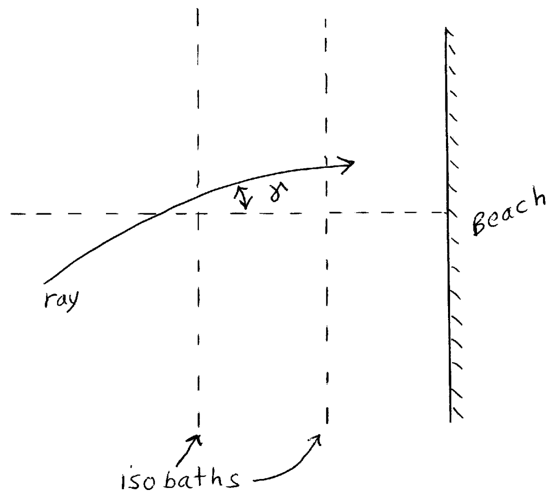

Figure 6.1: Refraction causes the rays to bend toward shore.

$$\sin\gamma = \frac{l}{\kappa} \tag{6.50}$$

where $\kappa = \sqrt{k^2 + l^2}$. The phase speed $c = \omega/\kappa$. It then follows from (6.50) that

$$\sin\gamma = \frac{l_0}{\omega_0}c, \tag{6.51}$$

a result known as *Snell's law*. According to (6.51), the decrease in phase speed $c(x)$ as waves enter shallow water must be accompanied by a decrease in the angle γ between the direction of wave travel and the perpendicular to the beach. The waves turn toward shallow water.

For the situation just considered, in which $H = H(x)$, the rays are a family of identical curves; they differ only by a constant displacement in the y-direction. However, real bathymetry always has a long-shore variation, $H = H(x, y)$. In the case of Scripps Beach, this variation is extreme. As shown in figure 6.2, deep canyons cut perpendicular to the shoreline offshore of the Beach and Tennis Club, and north of Scripps pier.

In the case of two-dimensional bathymetry, no shortcut analogous to (6.47) is possible. We must solve (6.44) and (6.45) to find the rays $(x(t), y(t))$, and to find how the wavenumber $(k(t), l(t))$ varies along each ray. If we can do this for enough rays, we will have a fairly complete picture of the wave field. But unlike in the case $H = H(x)$, every ray will be different. More

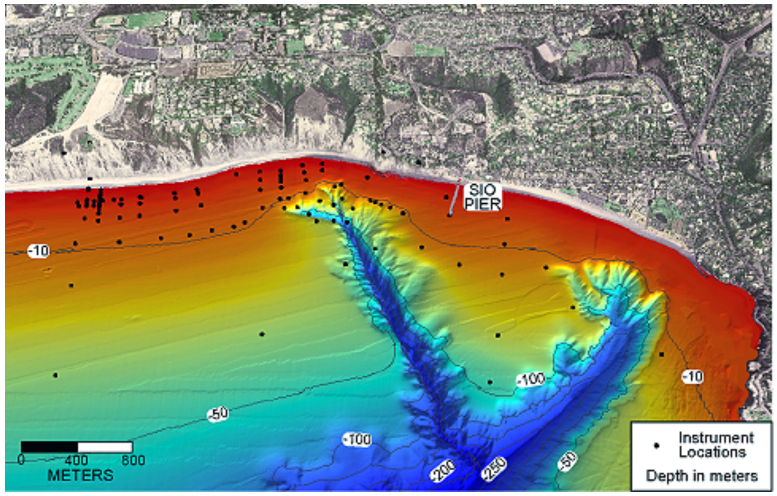

Figure 6.2: The two branches of Scripps Canyon near La Jolla, California.

work is required!

How do we do this? Take a large bathymetric chart and draw a line approximately parallel to the shore in deep water. At equal intervals along this line, make little ×'s, not too far apart. Each × is the beginning point for a ray. At each ×, k and l have the values determined by the incoming swell. Now solve (6.44) and (6.45) to determine the path of each ray and how the wavenumbers change along it. Perhaps it is easiest to think about (6.44) and (6.45) in their equivalent vector forms,

$$\frac{d\mathbf{x}}{dt} = \mathbf{c}_g(\mathbf{k}, H) \tag{6.52}$$

and

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial \Omega}{\partial H} \nabla H \tag{6.53}$$

Draw the line increment $\Delta \mathbf{x} = \mathbf{c}_g(\mathbf{k}, H)\Delta t$ using the starting values of $\mathbf{k}$ and H. Update H to the new location, and update $\mathbf{k}$ with the increment $\Delta \mathbf{k} = -(\partial\Omega/\partial H)\Delta t \nabla H$. Then repeat the process for another step. For greater accuracy, reduce the size of Δt. Of course, no one does this 'by hand' anymore; computers do all the work.

What does the solution actually look like? Remembering that $\mathbf{c}_g$ has the same direction as $\mathbf{k}$ at every location, and noting that the wavenumber increments dictated by (6.53) are always in the direction of $-\nabla H$, that is,

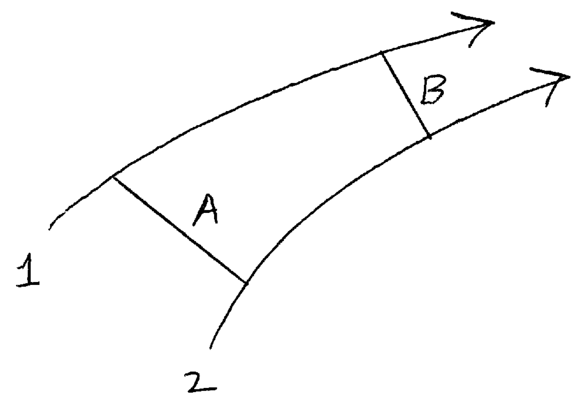

Figure 6.3: The energy flux between two rays is constant, hence the wave height is greater at section B than at section A.

toward shallower water, we see that the rays *always* bend toward shallow water. In deep water, the rays are parallel, corresponding to uniform incoming swell, but as the waves begin to feel the bottom, refraction bends the rays in the direction of most rapid shoaling. This means that rays are bent away from the canyons near the B&T and north of Scripps pier. These same rays converge—become closer together—in the area between the canyons.

Why do we care so much about rays? They are a handy way of calculating $\mathbf{k}$, but their real importance goes far beyond that. It lies in the energy equation, (6.43), which, in the case of steady waves, becomes

$$\nabla \cdot (\mathbf{c}_g E) = 0 \tag{6.54}$$

According to (6.54), the flux of energy is non-divergent. Consider the two nearby rays (labeled 1 and 2) and two cross-sections (labeled A and B) in figure 6.3. Apply the divergence theorem

$$\iint d\mathbf{x}\, \nabla \cdot \mathbf{F} = \oint ds\, \mathbf{F} \cdot \mathbf{n} \tag{6.55}$$

to the area bounded by the rays and cross-sections. Since $\mathbf{F} = \mathbf{c}_g E$ is tangent to the rays, only the cross sections contribute, and we conclude that

$$\int_A ds\, E\mathbf{c}_g \cdot \mathbf{n} = \int_B ds\, E\mathbf{c}_g \cdot \mathbf{n} \tag{6.56}$$

The flux of energy between any two rays is a constant. Suppose, as in the case $H = H(x)$, that the two rays always stay the same distance apart. As the

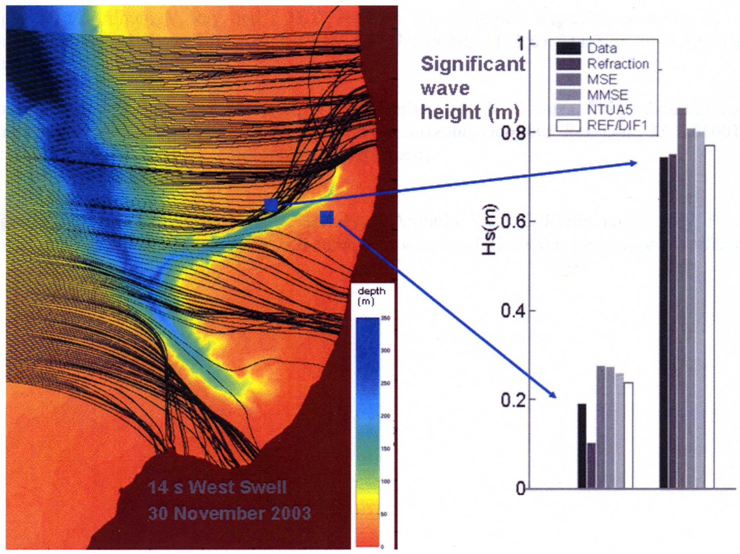

Figure 6.4: Wave refraction in Scripps Canyon.

rays enter shallow water, the group velocity decreases. The energy density E must therefore *increase*, to keep the flux constant. We have discussed this phenomenon several times before.

Now suppose that the two rays converge, as they actually do between the two canyons at La Jolla Shores. The energy density E now increases for an *additional* reason, besides the decrease in group velocity. As the rays converge—as they become closer together—E must increase to compensate the decrease in the distance between the rays. Thus, the energy flux diverted from the canyon areas by refraction gets concentrated in the area between the canyons. This is why the waves are low at the B&T, and why the surfing is good a bit further north.

Figure 6.4 shows the rays at La Jolla Shores corresponding to incoming swell with a period of 14 seconds. (This figure is from a 2007 paper by Magne et al in the *Journal of Geophysical Research*, courtesy of co-author Tom Herbers.) The figure shows how rays are refracted away from the two canyon heads, always bending toward shallow water. Where the rays converge, the wave energy is expected to be large. Where the rays diverge, the wave energy is expected to be small. Can you see why small children should swim near the B&T, and why the surfing is often good between the two canyon heads? The right panel shows the observed and predicted wave heights at two buoy locations.

In overall summary, you can solve the ray equations to get the ray paths and the variation of wavenumber along them. That tells you the wavelength and the direction of the shoaling wave field. To get the wave amplitude, you apply the principle that the energy flux between rays is a constant. Do all of this for enough rays, and you have the complete, slowly-varying wave field everywhere seaward of the breaker zone. Of course, all of this is based upon linear—small amplitude—theory. No matter how carefully you do it, errors increase as the amplitude of the shoaling waves increases. However, experience shows that things work pretty well right up to the point of wave breaking. That means that if you are willing to believe criteria like (2.46)—which seems to work on gently sloping beaches—you can even use ray theory to predict where the waves will break.

After that, all bets are off! Linear theory breaks down, and Postulates #1 and #2 no longer apply. A more complete description of fluid mechanics is required, and even with that more complete description, it turns out to be very difficult to understand exactly what is going on. The science of the surf zone is largely descriptive. To follow it as best we can, we need to invest some time in a more complete development of fluid mechanics. But before we do that, there is room for one more, unsurpassingly beautiful topic that lies almost wholly within the realm of Postulates #1 and #2. It is the pattern of waves generated by a steadily moving ship.

Chapter 7

Rogue waves and ship waves

In previous chapters we omitted the effects of ocean currents; we always assumed that the waves propagate in an ocean whose average velocity vanishes. Suppose, instead, that ocean currents are present. Let $\mathbf{U}(x, y, t)$ be the velocity of the current. We assume that $\mathbf{U}$ is horizontal, z-independent, and slowly varying in x, y, and t. More precisely, $\mathbf{U}$ changes by a negligible amount over the vertical decay-scale of the waves, and varies slowly, if at all, on the horizontal scale of the waves. That is, $\mathbf{U}(x, y, t)$ changes by only a small percentage in each wavelength or wave period.

If all these assumptions hold, then the waves ride on the current in the same way that you ride on a bus. The current speed simply adds to the phase speed. If the current flows in the same direction as the waves are propagating (with respect to the water), then the waves appear to be moving faster than normal to a stationary observer. All of this is summed up by the dispersion relation,

$$\omega = \mathbf{U} \cdot \mathbf{k} + \Omega(\mathbf{k}, H) \tag{7.1}$$

where $\mathbf{k} = (k, l)$ is the horizontal wavenumber; H is the ocean depth; and

$$\Omega(\mathbf{k}, H) = \sqrt{g\kappa \tanh(\kappa H)} \tag{7.2}$$

with $\kappa = \sqrt{k^2 + l^2}$ as before. Now, however, $\Omega(\mathbf{k}, H)$ is the *relative* frequency —the frequency measured by an observer drifting with the current—while ω is the frequency measured by a stationary observer. These two frequencies differ by the *Doppler shift* $\mathbf{U} \cdot \mathbf{k}$. If the waves are propagating in the same direction as the current, then $\mathbf{U} \cdot \mathbf{k} > 0$, and the stationary observer measures a higher frequency than the observer drifting with the current. If $\mathbf{U} \cdot \mathbf{k} < 0$, the reverse is true.

The results of chapter 6 generalize to the case in which ocean currents are present. It is simply a matter of replacing $\Omega(\mathbf{k}, H)$ by $\mathbf{U} \cdot \mathbf{k} + \Omega(\mathbf{k}, H)$. The two components of the prescribed velocity field $\mathbf{U}(x, y, t)$ are treated as 'medium parameters' like $H(x, y, t)$. The ray equations (6.44) generalize to

$$\frac{dx}{dt} = U + \left.\frac{\partial \Omega}{\partial k}\right|_{H,l}, \qquad \frac{dy}{dt} = V + \left.\frac{\partial \Omega}{\partial l}\right|_{H,k} \tag{7.3}$$

where $\mathbf{U} = (U, V)$, and the refraction equations (6.45) generalize to

$$\frac{dk}{dt} = -\left.\frac{\partial \Omega}{\partial H}\right|_{k,l} \frac{\partial H}{\partial x} - k\frac{\partial U}{\partial x} - l\frac{\partial V}{\partial x}, \qquad \frac{dl}{dt} = -\left.\frac{\partial \Omega}{\partial H}\right|_{k,l} \frac{\partial H}{\partial y} - k\frac{\partial U}{\partial y} - l\frac{\partial V}{\partial y} \tag{7.4}$$

and

$$\frac{d\omega}{dt} = \left.\frac{\partial \Omega}{\partial H}\right|_{k,l} \frac{\partial H}{\partial t} + k\frac{\partial U}{\partial t} + l\frac{\partial V}{\partial t} \tag{7.5}$$

In vector notation, (7.3) takes the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{U} + (\mathbf{c}_g)_{rel} \equiv \mathbf{c}_g \tag{7.6}$$

and (7.4) takes the form

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial \Omega}{\partial H}\nabla H - k\nabla U - l\nabla V \tag{7.7}$$

Here $(\mathbf{c}_g)_{rel}$ is the *relative* group velocity—the group velocity relative to the current—and $\nabla \equiv (\partial_x, \partial_y)$. Equations (7.6) and (7.7) generalize (6.52) and (6.53). In deep water, (7.6) and (7.7) reduce to

$$\text{DW} \qquad \frac{d\mathbf{x}}{dt} = \mathbf{U} + \left(\frac{k}{\kappa}, \frac{l}{\kappa}\right)\frac{1}{2}\sqrt{\frac{g}{\kappa}} \tag{7.8}$$

and

$$\text{DW} \qquad \frac{d\mathbf{k}}{dt} = -k\nabla U - l\nabla V \tag{7.9}$$

In deep water, refraction occurs only if the current $\mathbf{U}$ varies with location.

The energy equation (6.43) also requires generalization; it becomes

$$\frac{\partial}{\partial t}\left(\frac{E}{\Omega}\right) + \nabla \cdot \left(\mathbf{c}_g \frac{E}{\Omega}\right) = 0 \tag{7.10}$$

The quotient E/Ω, called *wave action*, replaces the energy as the important conserved quantity when ocean currents are present. We will not attempt to derive (7.10), which is somewhat difficult.

In chapter 6 we used the ray equations as a means of mapping out a slowly varying wavetrain, such as that which occurs when a basic swell wave approaches the beach. Here we want to mention a completely different—and equally useful—interpretation of (7.6) and (7.7). The ray equations (7.6-7) describe the evolution of a wave packet with 'carrier wavevector' $\mathbf{k}(t)$ located at $\mathbf{x}(t)$. According to (7.10), the action of the wave packet is conserved.

Refraction by ocean currents can cause the wave energy E to become very large. In fact, this is *one* explanation for the very rare, but very large, 'rogue waves' that occasionally damage or destroy large ships. How might this work? We consider a simple example in one horizontal dimension. Let

$$U(x) = \begin{cases} 0, & x < 0 \\ -sx, & x > 0 \end{cases} \tag{7.11}$$

be the velocity of the current, where s is a positive constant. The water is deep. A wave packet with wavenumber $k_0 > 0$ propagates toward positive x from the direction of $x = -\infty$. On $x < 0$ its wavenumber remains constant, and the wavepacket moves at the constant speed $\frac{1}{2}\sqrt{g/k_0}$. However, as soon as it passes $x = 0$, the wavepacket's location and wavenumber are determined by the ray equations (7.8-9) in the form

$$\frac{dx}{dt} = U + \frac{\partial \Omega}{\partial k} = -sx + \frac{1}{2}\sqrt{\frac{g}{k}} \tag{7.12}$$

and

$$\frac{dk}{dt} = -k\frac{\partial U}{\partial x} = sk \tag{7.13}$$

If we let $t = 0$ correspond to the time at which the wavepacket passes $x = 0$, then (7.13) tells us that

$$k(t) = k_0 e^{st} \tag{7.14}$$

The wavenumber grows exponentially in time; the wavelength gets shorter as the wavepacket is 'squeezed' by the increasingly strong, adverse current. What happens to the wavepacket's location? Substituting (7.14) into (7.12), we obtain

$$\frac{dx}{dt} + sx = \frac{1}{2}\sqrt{\frac{g}{k_0}}e^{-st/2} \tag{7.15}$$

which must be solved for $x(t)$. To do this, we note that (7.15) can be written in the form

$$e^{-st}\frac{d}{dt}\left(e^{st}x\right) = \frac{1}{2}\sqrt{\frac{g}{k_0}}e^{-st/2} \tag{7.16}$$

or,

$$\frac{d}{dt}\left(e^{st}x\right) = \frac{1}{2}\sqrt{\frac{g}{k_0}}e^{st/2} \tag{7.17}$$

which can be directly integrated. Choosing the integration constant so that $x = 0$ at $t = 0$, we obtain

$$x(t) = \frac{1}{s}\sqrt{\frac{g}{k_0}}\left(e^{-st/2} - e^{-st}\right) \tag{7.18}$$

for the location of the wavepacket. The wavepacket reaches its furthest penetration toward positive x at the moment when $dx/dt = 0$. As you can easily show, this occurs at the time $t = \ln 4/s$. At this time, $k = 4k_0$; the wavelength has been squeezed to one fourth its original value. Thereafter the wavepacket is swept backward by the current, approaching, but never reaching, the point $x = 0$. All the while, its wavenumber continues to grow exponentially, according to (7.14).

The physical interpretation of this solution is as follows. The continual, exponential growth of the wavenumber causes the *relative* group velocity—the velocity of the wave packet with respect to the water—to continually decrease until it is overpowered by the oncoming current. Thereafter, the wavepacket is carried backward by the current. The wavepacket never stops fighting the current, but it is an increasingly ineffective fighter, because its relative group velocity just keeps on decreasing.

More interesting than the location of the wavepacket is what happens to its energy. The relative frequency $\Omega = \sqrt{gk}$ of the wavepacket grows exponentially in time, because the wavenumber k grows exponentially in time. However, the wave action E/Ω is conserved. This is only possible if E too grows exponentially. The wavepacket gradually acquires an *infinite* energy. Of course this is only an indication of what really happens. Linear theory—based on small-amplitude waves—breaks down long before the waves get very large. Nevertheless, we have a definite prediction that waves encountering an adverse current increase in size. Where does all this wave energy come from? It comes from the current itself. Refraction by currents causes waves to grow by transferring energy from the ocean current to the wave.

The example we have worked out is a very simple one. The two-dimensional case offers many more possibilities. For example, in two dimensions ocean currents can cause rays to converge, thereby focusing wave energy. However, some people think that rogue waves have nothing to do with ocean currents at all, because giant waves sometimes occur where currents are small. Instead, they talk about 'nonlinear self-focusing of waves.' (If you know what that means, you know too much to be taking this course.) Nevertheless, there is one place where currents are strong candidates for creating rogue waves. The Agulhas Current flows east to west around the southern tip of Africa. There it encounters very strong westerly winds that generate waves moving in the opposite direction. Dozens of oil tankers have been damaged by giant waves in this area. Elsewhere, measurements from ships, oil platforms, and now satellite SARs show that 100-foot waves are regular if uncommon occurrences. By one estimate, about 10 such waves are present somewhere in the world ocean at any one time.

From waves that are rarely observed, we now turn to waves that are seen all the time. We use the theory of slowly varying waves to explain the beautiful but rather complex pattern of waves that occur in the wake of a steadily moving ship.

Let the ship move at constant speed U_0 in the negative x-direction. We shall view the situation in a reference frame that is moving with the ship. In this reference frame, the ship is stationary at, say, $(x, y) = (0, 0)$, and a uniform current flows in the *positive* x-direction at speed U_0. Assuming that the ship is in deep water, the dispersion relation (7.1) becomes

$$\omega = U_0 k + \sqrt{g\kappa} \tag{7.19}$$

where, once again, $\kappa = \sqrt{k^2 + l^2}$. The corresponding group velocity is

$$\mathbf{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l} \right) = \left(U_0 + \frac{k}{2\kappa}\sqrt{\frac{g}{\kappa}}, \frac{l}{2\kappa}\sqrt{\frac{g}{\kappa}} \right) \tag{7.20}$$

In this reference frame, the wave field is not merely steady; it is *static*. That is, not only do the wavevector components $k(x, y)$ and $l(x, y)$ *not* depend on time; the frequency ω vanishes everywhere. This is because, in linear theory, the waves created by a time-independent forcing—the ship—must be time-independent themselves. To an observer standing in the stern of the ship, the wavy surface of the ocean appears to be a frozen surface.

Wait, you say. You have stood in the stern, and it is not like that. There is plenty of sloshing around. However, much of the time-dependence you see is caused by waves that would be there even if the ship were not. Then, too, the ship is not perfectly steady. It pitches and rolls in response to the waves. Finally, there is the usual limitation of linear theory: it applies only to waves with infinitesimal amplitude. Real ship waves can be large. They can even be large enough to break. All of these non-static effects are most important very close to the ship. This is where the waves are being generated, and where their energy is most concentrated. But farther out in the wake, the static pattern of waves locked to the ship is quite striking. It is this pattern that we seek to explain with linear theory.

If the frequency vanishes, then (7.19) becomes

$$0 = U_0 k + \sqrt{g\left(k^2 + l^2\right)^{1/2}} \tag{7.21}$$

which is one equation in the unknown, steady wavenumbers $k(x, y)$ and $l(x, y)$. We need a second equation. The second equation may be deduced from the fact that the ship itself is the source of all the wave energy in its wake. From a distance, the ship looks like a point source of energy at $x = y = 0$. This energy moves outward on rays given by

$$\frac{dx}{dt} = U_0 + \frac{1}{2}\frac{k}{\kappa}\sqrt{\frac{g}{\kappa}}, \qquad \frac{dy}{dt} = \frac{1}{2}\frac{l}{\kappa}\sqrt{\frac{g}{\kappa}} \tag{7.22}$$

On each ray,

$$\frac{dk}{dt} = \frac{dl}{dt} = 0 \tag{7.23}$$

Because there is no refraction, the wavevector components are constant along each ray. It follows that each ray is a straight line passing through the origin with slope

$$\frac{y}{x} = \frac{\frac{1}{2}\frac{l}{\kappa}\sqrt{\frac{g}{\kappa}}}{U_0 + \frac{1}{2}\frac{k}{\kappa}\sqrt{\frac{g}{\kappa}}} \tag{7.24}$$

which can also be written in the simpler form

$$\frac{y}{x} = \frac{l}{2\kappa\sqrt{\frac{\kappa}{g}}U_0 + k} \tag{7.25}$$

For given, arbitrary x and y, we solve (7.21) and (7.25) for $k(x, y)$ and $l(x, y)$. That gives the wave pattern in the wake of the ship.

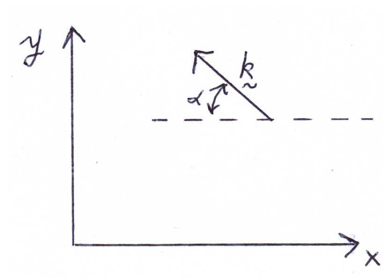

Figure 7.1: The angle α is the angle between the wavevector $\mathbf{k}$ and the direction in which the ship is moving.

To solve these equations, substitute (7.21) into (7.25) to obtain

$$\frac{y}{x} = \frac{lk}{-2\kappa^2 + k^2} \tag{7.26}$$

Let

$$\mathbf{k} \equiv (k, l) = \kappa(-\cos\alpha, \sin\alpha) \tag{7.27}$$

where α is the angle between $\mathbf{k}$ and the direction in which the ship is moving (figure 7.1). Then (7.26) becomes an equation in α alone, namely

$$\frac{y}{x} = \frac{\sin\alpha\cos\alpha}{2 - \cos^2\alpha} \tag{7.28}$$

Using the trigonometric identity $\cos^2\alpha = 1/(1 + \tan^2\alpha)$, we re-write this as

$$\frac{y}{x} = \frac{\tan\alpha}{1 + 2\tan^2\alpha} \tag{7.29}$$

Given x and y, we first find α as the solution to (7.29). Then we find κ from

$$\kappa = \frac{g}{{U_0}^2\cos^2\alpha} \tag{7.30}$$

which follows from (7.21). Then $\mathbf{k}$ is given by (7.27).

It is best to regard the right-hand side of (7.29) as a function of $\tan\alpha$ (figure 7.2). The range of α is from 0 to $\pi/2$. That is, the x-component of $\mathbf{k}$ must be negative; only then does the wave point against the current; only then can the two terms in (7.21) balance. It follows that the range of $\tan\alpha$ is from 0 to ∞. In that range, the right-hand side of (7.29) increases from 0 to

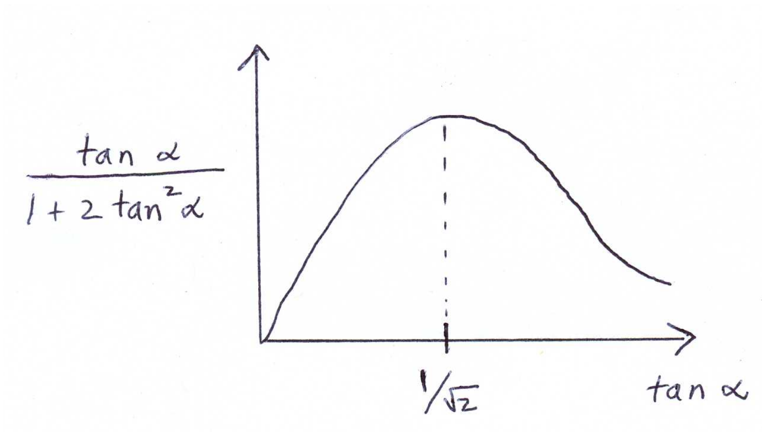

Figure 7.2: The ratio y/x as a function of $\tan\alpha$.

a maximum at $\tan\alpha = 1/\sqrt{2}$, and then asymptotes to 0 as $\tan\alpha$ approaches ∞. At the maximum,

$$\frac{y}{x} = \frac{1}{2\sqrt{2}} = \tan(19.5^{\text{o}}) \tag{7.31}$$

For greater y/x than this, (7.29) has no solution. Thus the ship's wake lies entirely within a wedge with angle $2 \times 19.5^{\text{o}} = 39^{\text{o}}$, regardless of the ship's speed. Outside this wedge—that is, for y/x greater than (7.31)—the sea surface is undisturbed. See figure 7.3.

At the edge of the wake, (7.29) has the single solution $\alpha = \tan^{-1}(1/\sqrt{2}) = 35.3^{\text{o}}$. By (7.30) this corresponds to

$$\kappa = \frac{g}{{U_0}^2(2/3)} \tag{7.32}$$

Thus, at the edge of the wake, the wavenumbers are

$$(k, l) = \frac{3g}{2{U_0}^2}\left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{2}}\right) \tag{7.33}$$

If you water-ski, this is the wave you hop over to get outside the wake.

For all y/x less than the maximum, (7.29) has two solutions. Thus, at any point within the wake there are two distinct waves present. At $y = 0$, directly behind the ship, the two solutions are $\alpha = 0$ and $\alpha = \pi/2$. The

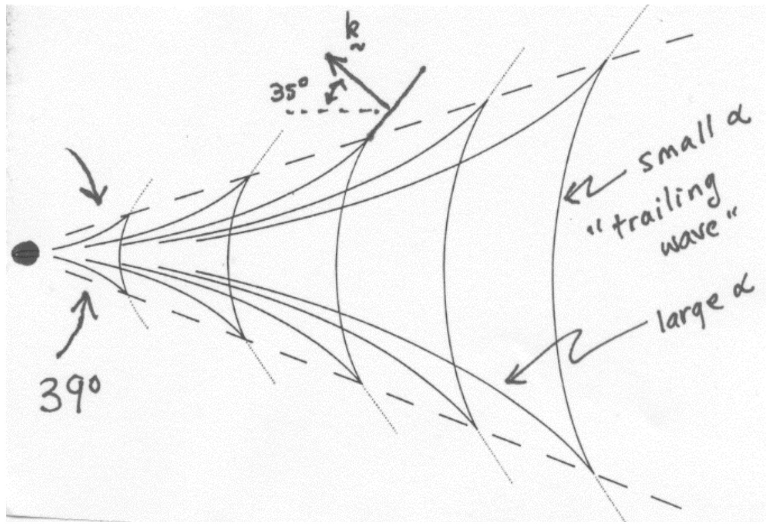

Figure 7.3: Two waves are present at all locations within the wake.

solution $\alpha = 0$ corresponds to a wavevector pointing directly at the ship. By (7.30) this wave has wavelength

$$\lambda_{max} = \frac{2\pi}{\kappa_{min}} = \frac{2\pi {U_0}^2}{g} \tag{7.34}$$

which is the longest wavelength present anywhere in the wake. The solution $\alpha = \pi/2$ corresponds to a wavevector pointing at right angles to the line $y = 0$, and to an infinitesimal wavelength. As y/x increases, the small-α solution and the large-α solutions converge to the single solution (7.33) at the edge of the wake.

We have found $\mathbf{k}(x, y)$; we know the wavevectors everywhere inside the wake. But what do the waves really look like? Recall the general expression for the slowly varying wavetrain,

$$\eta = A(x, y) \cos\theta(x, y) \tag{7.35}$$

A wave crest corresponds to a curve along which

$$\theta(x, y) = 2\pi n, \quad n = 0, 1, 2, \ldots \tag{7.36}$$

A wave trough corresponds to

$$\theta(x, y) = \pi + 2\pi n \tag{7.37}$$

Thus, to get a picture of the wake, we must determine the lines of constant θ. We must use our knowledge of $k = \partial\theta/\partial x$ and $l = \partial\theta/\partial y$ to find $\theta(x, y)$. Now,

$$\theta(x, y) - \theta(x_0, y_0) = \int_{\mathbf{x}_0}^{\mathbf{x}} \nabla\theta \cdot d\mathbf{x} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{k} \cdot d\mathbf{x} \tag{7.38}$$

where the integral is along *any* path between $\mathbf{x}_0$ and $\mathbf{x}$. We arbitrarily choose $\mathbf{x}_0 = (0, 0)$. Although we can choose any path between $(0, 0)$ and the arbitrary point (x, y) to evaluate the last integral in (7.38), it is most convenient to use the straight line of constant slope y/x, because, as we already know, both components of $\mathbf{k}$ are constant along such a line. Thus (7.38) becomes

$$\theta(x, y) - \theta(x_0, y_0) = kx + ly \tag{7.39}$$

But since (7.39) holds along every such line, we may write

$$\theta(x, y) - \theta(x_0, y_0) = k(x, y)x + l(x, y)y \tag{7.40}$$

where $k(x, y)$ and $l(x, y)$ are the previously determined wavevector components. Since $k(x, y)$ and $l(x, y)$ as known functions, we may regard (7.40) as an equation for the relative phase, $\Delta\theta \equiv \theta(x, y) - \theta(x_0, y_0)$, as a function of x and y. Alternatively, we may regard

$$k(x, y)x + l(x, y)y = \Delta\theta \tag{7.41}$$

as the equation for a line of constant phase, where the right-hand side is a constant. For given, constant $\Delta\theta$, (7.41) implicitly defines a line $y(x)$ that is locally parallel to wave crests.

What does such a line look like? Substituting (7.27) and (7.30) into (7.41) yields

$$\frac{g}{{U_0}^2 \cos^2\alpha}(-x\cos\alpha + y\sin\alpha) = \Delta\theta \tag{7.42}$$

But α is itself a function of x and y, implicitly determined by (7.30). Solving (7.42) and (7.29), and simplifying a bit, we obtain the equation for our line of constant phase in the parametric form

$$x = -\frac{\Delta\theta\, {U_0}^2}{g}\cos\alpha(1 + \sin^2\alpha) \tag{7.43a}$$

and

$$y = -\frac{\Delta\theta\, {U_0}^2}{g}\cos^2\alpha\sin\alpha \tag{7.43b}$$

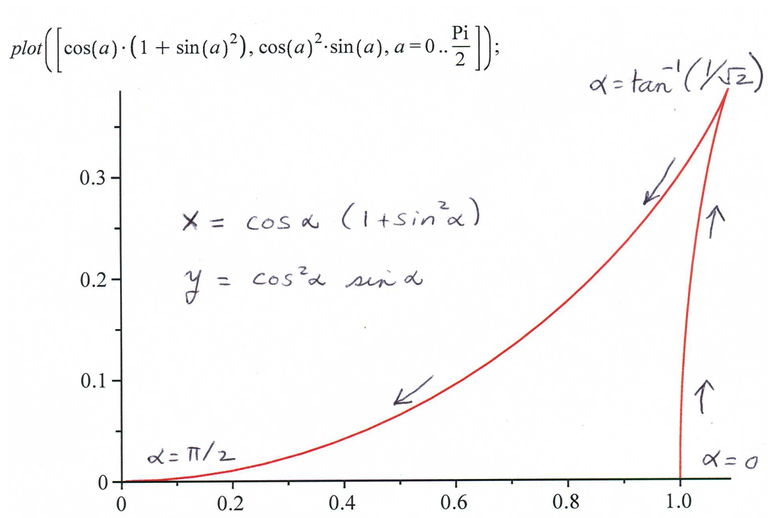

Figure 7.4: The wave crests have the universal shape determined by eqns (7.45).

As α runs through its range from 0 to $\pi/2$, the equations (7.43) trace out a line locally parallel to wave crests. If we agree to measure x and y in units of

$$-\Delta\theta\, {U_0}^2/g \tag{7.44}$$

then (7.43) assume the universal form,

$$x = \cos\alpha(1+\sin^2\alpha), \qquad y = \cos^2\alpha\sin\alpha \tag{7.45}$$

The apparent sign of (7.44) is of no concern, because phase is always arbitrary by $\pm 2\pi n$. Only the *size* of (7.44) has physical importance, and it merely controls the distance between successive crests, greater for larger ship speeds. The crests themselves have the universal shape (7.45). Set $\alpha = 0$ in (7.45) and you have put yourself directly behind the ship at $(x, y) = (1, 0)$. As α increases to $\tan^{-1}(1/\sqrt{2})$ you move outward, and slightly backward, along the crest of the long-wavelength 'trailing wave', to the cusp at the edge of the wake. See figure 7.4. Then, as α increases from $\tan^{-1}(1/\sqrt{2})$ to $\pi/2$, you move rapidly toward the ship, along the crest of the short-wavelength wave. The wave pattern reveals itself! The algebra is somewhat tedious, but the underlying ideas are quite simple: Postulates #1 and #2, and the idea of a slowly varying wavetrain.

We have said nothing at all about the slowly varying amplitude $A(x, y)$. If only we knew its value in the ray tubes originating at the ship, we could determine $A(x, y)$ throughout the wake, following the same procedure as in chapter 6. But this is a very hard problem! To solve it, naval architects use powerful computers to solve the general equations governing the ship and the surrounding fluid—equations that we have not yet even discussed. The problem is important because the energy in the waves generated by the ship represents a sizable fraction of the energy produced by the ship's engines. For this reason, ship hulls are designed to minimize wave generation.

Although the complete problem is a difficult one, we can get some idea about $A(x, y)$ by using the following principle: Ships tend to generate waves with wavelengths that are about the same size as the ship itself. Let L be the length of the ship. We have seen that the longest waves in the wake are the 'trailing waves'—the waves with crests nearly perpendicular to the direction of ship motion. According to (7.32) and (7.34), their wavelengths vary between $\lambda_{max} = 2\pi {U_0}^2/g$ directly behind the ship to $2\lambda_{max}/3$ at the edge of the wake. Roughy speaking, then, these 'trailing waves' all have wavelengths of order ${U_0}^2/g$. In contrast, the other family of waves—those with crests more nearly parallel to the ship track—have wavelengths shorter than ${U_0}^2/g$. In fact, their wavelengths approach zero in the region directly behind the ship. It follows that if the ship's length L is large compared to ${U_0}^2/g$, then it generates mainly 'trailing waves'. If, on the other hand, L is small compared to ${U_0}^2/g$, then most of the wave energy is concentrated in the short waves with crests more nearly parallel to the ship's direction. If L is *much* smaller than ${U_0}^2/g$, then the most prominent waves are the waves with crests that nearly coincide with the axis of the wake. We can say this somewhat more succinctly by defining the *Froude number*,

$$Fr \equiv {U_0}^2/gL \tag{7.46}$$

If $Fr \ll 1$, then the ship mainly generates 'trailing waves' with crests nearly perpendicular to the wake axis. A big, slow-moving tugboat would be a good example of this. However, a short, fast speedboat with $Fr \gg 1$ mainly generates the shorter waves with crests nearly parallel to the wake axis.

You can find lots of pictures of ship wakes. In most of them, the wake is actually produced by a ship. However, in the remarkable satellite photo shown in figure 7.5, the "ship wake" is produced by air flowing over a volcanic island in the Indian Ocean. The island, located in the lower left of the

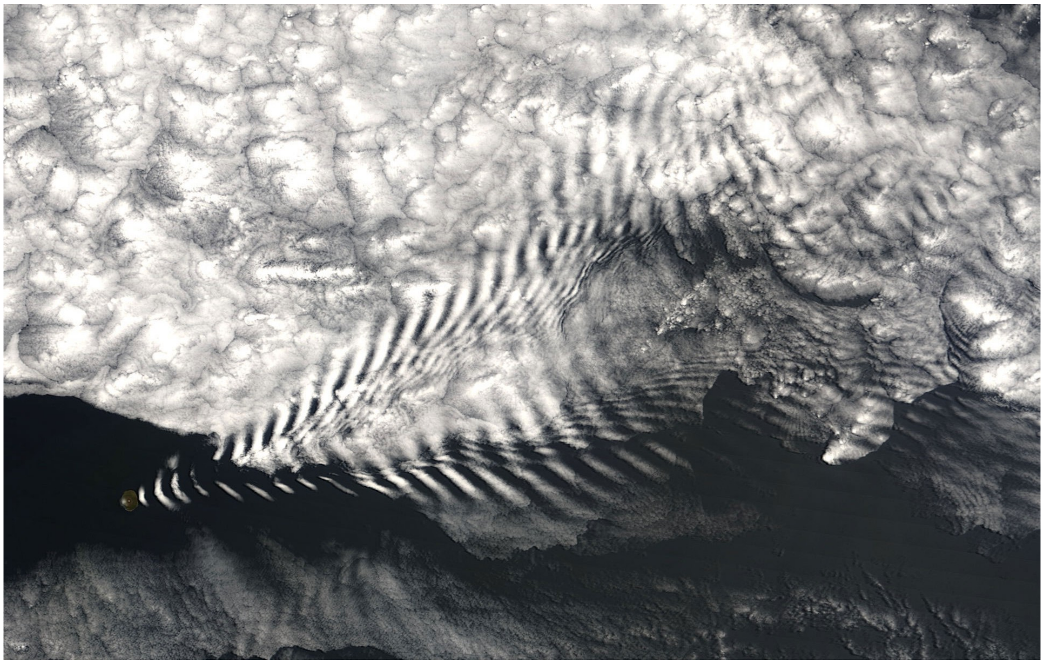

Figure 7.5: The cloud pattern produced by wind flowing past a volcanic island resembles our calculation of a ship wake.

photo, generates a wake of internal gravity waves in the atmosphere. In these internal waves, clouds form where the air rises and clouds evaporate where the air descends. Thus the cloud pattern in the photograph reveals the pattern of vertical velocity in the atmosphere. In this photo the wind blows from left to right. As the wind blows over the island, air is forced upward, and its water vapor condenses into droplets. This explains the “cloud spot” on the windward slope of the volcano. As the air descends on the leeward slope, the cloud evaporates. But air, like water, has inertia. The updrafts and downdrafts “overshoot”, generating a field of internal gravity waves. In the far field, the island looks like a point source, just as in our calculation of the ship wake.

The really surprising thing is this: Our calculation used the deep-water dispersion relation for water waves. However, the physics of atmospheric internal gravity waves is much more general and more complex. For one thing, it depends upon the mean temperature profile of the atmosphere, which varies from place to place and from time to time. Why, in this case, is the atmosphere responding to the island in the same way that the surface of the ocean responds to a ship?

On this particular day, a thick layer of cold air must have been present at the bottom of the atmosphere, just above the ocean surface. The waves in the photo probably represent undulations in the upper boundary of this thick layer of cold air. Without knowing the wind speed, the air temperature, and the thickness of the cold-air layer, it is impossible to apply a quantitative test. However, the angle of the wake in the photo is very close to that predicted by our theory. Try measuring it yourself!

Chapter 8

Hydrodynamics and linear theory

Up to now, we have used a fair amount of math and a very little physics. Our physics has consisted of Postulates #1 and #2, and a rather innocent assumption about wave energy, namely, that the energy was proportional to the square of the wave amplitude. In the previous chapter we used the conservation of wave action; you were asked to take that on faith.

Without more physics, we can't go much further. We need the general equations of fluid mechanics. In this chapter, we derive those equations, and we use them to justify Postulates #1 and #2. In chapters 9 and 10 we investigate some phenomena that Postulates #1 and #2 can't explain.

Fluid mechanics is a *field theory*, like electrodynamics. In electrodynamics, the fundamental fields are the electric field $\mathbf{E}(x, y, z, t)$ and the magnetic field $\mathbf{B}(x, y, z, t)$. The fields depend continuously on location and time. The field equations are Maxwell's equations. Maxwell's equations require no derivation. Within the context of classical physics, they represent fundamental physical law.

In fluid dynamics, the fields include the mass density $\rho(x, y, z, t)$, the pressure $p(x, y, z, t)$, and the fluid velocity $\mathbf{v}(x, y, z, t)$. However, the fluid equations do not represent fundamental physical law in the same way that Maxwell's equations do. The fluid equations are merely an approximation to a deeper reality. That deeper reality is—again within the context of classical physics—molecules whizzing around and occasionally colliding, all the while obeying $F = ma$. The mass density $\rho(x, y, z, t)$ is simply the *average* mass per unit volume of all the molecules in the vicinity of (x, y, z) at time t. It is a smooth function of its arguments solely because of the way it is defined. Similar remarks apply to the other fluid fields.

The absolutely best way to derive the fluid equations is to average over the equations governing the molecules, principally $F = ma$. That turns out to be a big project. Instead, most fluids books derive the fluid equations by *pretending* that the fluid is a *continuum*—a continuous distribution of mass in space—and imagining how such a thing would behave if it actually existed. We will do that too.

Our derivation of the fluid equations leans heavily on the idea of a conservation law. This is a concept that we have already encountered several times. Let n be the amount of X per unit volume. If the total amount of X is conserved, then n must obey an equation of the form

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{F} = 0 \tag{8.1}$$

where $\mathbf{F} = (F_x, F_y, F_z)$ is the *flux* of X. Equivalently,

$$\frac{\partial n}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0 \tag{8.2}$$

If the X moves at the velocity $\mathbf{q}$, then $\mathbf{F} = n\mathbf{q}$, and (8.1) takes the form

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{q}) = 0 \tag{8.3}$$

In previous chapters, n was the wave energy E, which travels at the group velocity; hence $\mathbf{q} = \mathbf{c}_g$.

The density ρ is the amount of mass per unit volume. The mass moves at the velocity $\mathbf{v} = (u, v, w)$ of the fluid. Therefore, the conservation law for mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \tag{8.4}$$

The momentum per unit volume is $\rho\mathbf{v} = \rho(u, v, w)$. We expect a conservation law for each of its components. We start by considering ρu, the x-component of the momentum per unit volume. If the momentum were merely carried around by the fluid in the same way as mass, then we would take $n = \rho u$ and $\mathbf{q} = \mathbf{v}$ to get

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u\mathbf{v}) = 0 \quad \text{WRONG} \tag{8.5}$$

Equation (8.5) ignores the fact that *forces* cause the momentum to change. Thus, instead of (8.5), we must write

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{v}) = f_x \quad \text{RIGHT} \tag{8.6}$$

where $\mathbf{f}$ is the force per unit volume and f_x is its x-component. In fluid mechanics, there are several possible forces, but one is *always* present. That is the pressure force, the force of the fluid on itself.

"Slice" the fluid at any angle. The fluid on each side of your slice is pushing against the other side of the slice with a force per unit area equal to the pressure p. The direction of the force is always normal to the slice. In that sense the pressure is isotropic. By considering a small cubic volume of fluid, we see that the pressure force per unit volume is $\mathbf{f} = -\nabla p$. You can remember the sign by remembering that the fluid is being pushed away from where the pressure is high. Thus (8.6) becomes

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{v}) = -\frac{\partial p}{\partial x} \tag{8.7a}$$

Similarly, for the y-component of the momentum equation we have

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{v}) = -\frac{\partial p}{\partial y} \tag{8.7b}$$

The z-component—the vertical component—of the momentum equation contains an additional term, the force per unit volume caused by gravity. Including it, we have

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{v}) = -\frac{\partial p}{\partial z} - \rho g \tag{8.7c}$$

An important simplification occurs if the density is constant. The fluid equations (8.4) and (8.7) reduce to

$$\nabla \cdot \mathbf{v} = o \tag{8.8}$$

and

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u &= -\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v &= -\frac{\partial p}{\partial y} \\ \frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w &= -\frac{\partial p}{\partial z} - g \end{aligned} \tag{8.9}$$

where we have absorbed the constant density into the pressure. (New p equals old p divided by constant ρ.) We can write (8.9) as a single *vector* equation in the form

$$\frac{D\mathbf{v}}{Dt} = -\nabla(p + gz) \tag{8.10}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{8.11}$$

The operator (8.11) is often called the *advective derivative* or the *substantial derivative*. It is the time derivative measured by an observer moving at the fluid velocity. Equations (8.8) and (8.10) are the equations of *ideal hydrodynamics*. They represent 4 equations in the 4 unknowns p, u, v, and w.

Hydrodynamics is the branch of fluid mechanics that deals with constant-density fluids. The theory of water waves lies wholly within hydrodynamics. The equations (8.8, 8.10) are called *ideal* because they omit *viscosity*, the frictional force of the fluid rubbing against itself. If viscosity is included, then (8.10) generalizes to

$$\frac{D\mathbf{v}}{Dt} = -\nabla(p + gz) + \nu\nabla^2\mathbf{v} \tag{8.12}$$

where ν is the viscosity coefficient (equal to about $0.01\text{cm}^2\text{sec}^{-1}$ in water) and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{8.13}$$

is the Laplacian operator. The combination (8.8) and (8.12) is called the *Navier-Stokes equations*. However, since none of the phenomena we discuss involve viscosity in an essential way, we shall use ideal hydrodynamics. Thus, our fundamental equations are (8.8) and (8.10).

These equations take a simpler form if the velocity takes the special form

$$\mathbf{v} = (u, v, w) = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \tag{8.14}$$

This really *is* a special form. In the general case, $u(x, y, z, t)$, $v(x, y, z, t)$, and $w(x, y, z, t)$ are three independent fields. However, under the assumption

(8.14), all three components of velocity are determined by the *single* scalar field $\phi(x, y, z, t)$. As consequences of (8.14) we have

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} \tag{8.15}$$

These are special relations that are not satisfied by an arbitrary velocity field $\mathbf{v}(x, y, z, t)$. Flow satisfying (8.14) is called *potential flow*; ϕ is called the *velocity potential*. Before we explain *why* the velocity should take the special form (8.14), we show how much it simplifies our equations.

If the velocity field satisfies (8.14) then (8.8) reduces to *Laplace's equation*

$$\nabla^2 \phi = 0 \tag{8.16}$$

and (8.9a) takes the form

$$\frac{\partial}{\partial t}\frac{\partial \phi}{\partial x} + \nabla\phi \cdot \nabla\left(\frac{\partial \phi}{\partial x}\right) = -\frac{\partial p}{\partial x} \tag{8.17}$$

which is equivalent to

$$\frac{\partial}{\partial t}\frac{\partial \phi}{\partial x} + \nabla\phi \cdot \frac{\partial}{\partial x}\nabla\phi = -\frac{\partial p}{\partial x} \tag{8.18}$$

and to

$$\frac{\partial}{\partial t}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}\nabla\phi \cdot \nabla\phi\right) = -\frac{\partial p}{\partial x} \tag{8.19}$$

and finally to

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + p\right) = 0 \tag{8.20a}$$

From (8.9b) by similar steps we get

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + p\right) = 0 \tag{8.20b}$$

and from (8.9c) we get

$$\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + p + gz\right) = 0 \tag{8.20c}$$

We may write (8.20) as a single equation,

$$\nabla\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+p+gz\right)=0 \tag{8.21}$$

From (8.21) we conclude that

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+p+gz=C(t) \tag{8.22}$$

where $C(t)$ is a function only of time. This function turns out to be completely unnecessary. One way to see this is to note that we can absorb it into the first term in (8.22) by replacing

$$\phi\to\phi+\int C(t)\,dt \tag{8.23}$$

This replacement has no effect on the velocity (8.14). With this final simplification, the equations of ideal hydrodynamics become

$$\nabla^2\phi=0 \tag{8.24a}$$

and

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+p+gz=0 \tag{8.24b}$$

Equation (8.24b) is usually called the *Bernoulli equation*. Although much simpler than (8.8) and (8.10), (8.24) are by no means easy to solve. A principal difficulty is the quadratic term $\nabla\phi\cdot\nabla\phi$ in (8.24b).

Equations like (8.24b), which contain *products* of the dependent variables, are said to be *nonlinear*. Nonlinear partial differential equations are usually very difficult to solve. All of the mathematical methods we have used—and almost all of the methods you will learn as undergraduates—apply only to linear equations. There are few general methods for solving nonlinear equations, and most of those involve severe approximations. Nonlinear equations are very challenging! Yet, without the nonlinearity in (8.24b), waves would not steepen and break. Without the more complicated nonlinearity in (8.10), fluid flow would never be turbulent. The nonlinear property of the fluid equations is another property that sets them apart from Maxwell's equations. Maxwell's equations are linear, hence their solutions are very polite and well behaved.

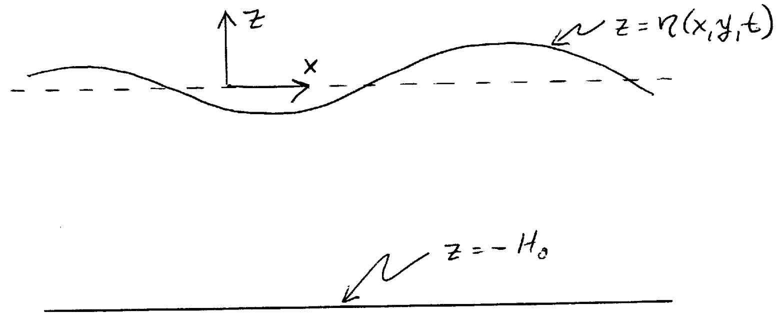

Figure 8.1: Geometry for the fundamental problem of water wave theory.

Now we pose the fundamental problem of water wave theory. We consider a horizontally unbounded ocean with a flat bottom at $z = -H_0$ and a free surface at $z = \eta(x, y, t)$ (figure 8.1). The fluid obeys the fundamental equations (8.24) plus boundary conditions at its top and bottom. The bottom boundary condition is that

$$w = \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -H_0 \tag{8.25}$$

because there can be no flow through the ocean bottom. At the top boundary there are two boundary conditions. The *kinematic boundary condition* states that fluid particles on the free surface must remain there,

$$\frac{D}{Dt}(z - \eta(x, y, t)) = w - \frac{D\eta}{Dt} = \frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} - \nabla\phi \cdot \nabla\eta = 0 \quad \text{at } z = \eta(x, y, t) \tag{8.26}$$

The *dynamic boundary condition* states that the pressure must be continuous across the free surface. Thus

$$p = p_a \quad \text{at } z = \eta(x, y, t) \tag{8.27}$$

where p_a is the pressure of the atmosphere. Condition (8.26) says that the free surface must move at the same velocity as the water particles that are on it; condition (8.27) forbids the pressure discontinuity that would result in an infinite acceleration of the free surface.

Now we collect all our equations and their boundary conditions. The equations are

$$\nabla^2 \phi = 0 \quad \text{on } -H_0 < z < \eta(x, y, t) \tag{8.28a}$$

and

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + p + gz = 0 \quad \text{on } -H_0 < z < \eta(x,y,t) \tag{8.28b}$$

and the boundary conditions are

$$\frac{\partial \phi}{\partial z}(x,y,-H_0,t) = 0, \tag{8.29a}$$

$$\frac{\partial \phi}{\partial z}(x,y,\eta(x,y,t),t) - \frac{\partial \eta}{\partial t} - \nabla\phi(x,y,\eta(x,y,t),t) \cdot \nabla\eta(x,y,t) = 0, \tag{8.29b}$$

and

$$p(x,y,\eta(x,y,t),t) = p_a(x,y,t) \tag{8.29c}$$

We have written the boundary conditions with full arguments to emphasize how complicated there are. The boundary conditions (8.29b,c) involve the unknowns ϕ and p evaluated at a location that *depends* on η. In other words, you need to know η before you can solve the equations that tell you what it is. Things can't get much worse than this! This problem disappears if we *linearize* (8.28) and (8.29) about the state of rest. But before we do this, we need to say something more about (8.29c).

In (8.29c) p_a is the atmospheric pressure at the free surface. Where do you get *that*? Well you could *solve* for it, by extending the problem to include the atmospheric flow. Then p_a just becomes another unknown field. More simply, you could *specify* $p_a(x,y,t)$, perhaps using some real measurements of atmospheric pressure. Or, you could do what we are going to do and just *ignore* p_a. That would be a poor idea if your goal is to understand how waves are generated; p_a is the thing that generates the waves. But if your goal is to study free waves, then you may be justified in ignoring p_a. Setting $p_a = 0$ corresponds to replacing the atmosphere by a vacuum. This is reasonable, because the mass density of the atmosphere is about 1000 times less than that of the ocean. To a first approximation, the ocean sees the atmosphere as a vacuum.

The state of rest corresponds to $\phi = \eta = 0$; the free surface is flat, and the velocity field vanishes. Suppose that waves are present but very small. That is, suppose that the solution to (8.28) and (8.29) represents a slight departure from the state of rest. Then ϕ and η are very small, and in equations like (8.28b), we may neglect terms proportional to the *products* of ϕ and η in comparison to terms that contain only one factor of ϕ or η. This is because small numbers get even smaller if you multiply them together.

For example, consider the boundary condition (8.29b). To expose the sizes, we perform a Taylor expansion of each term with respect to its third argument. For example,

$$\begin{aligned}\frac{\partial \phi}{\partial z}(x,y,\eta(x,y,t),t) &= \frac{\partial \phi}{\partial z}(x,y,0,t) + \frac{\partial^2 \phi}{\partial z^2}(x,y,0,t)\eta(x,y,t) + \cdots \\ &\approx \frac{\partial \phi}{\partial z}(x,y,0,t) \end{aligned} \tag{8.30}$$

because *products* of small terms are negligible compared to terms that are *linear* in ϕ or η. Thus, the linear approximation to (8.29b) is

$$\frac{\partial \phi}{\partial z}(x,y,0,t) = \frac{\partial \eta}{\partial t}(x,y,t) \tag{8.31}$$

As for (8.29c), we take $p_a = 0$ (as discussed above), so that the boundary condition becomes $p = 0$ at the free surface $z = \eta(x,y,t)$. To express this boundary condition in terms of our fundamental variables ϕ and η, we use (8.28b), which holds throughout the fluid, and we apply it at the free surface with $p = 0$. Then, linearizing this boundary condition in the same way as (8.28a), we obtain

$$\frac{\partial \phi}{\partial t}(x,y,0,t) + g\eta(x,y,t) = 0 \tag{8.32}$$

as the linear approximation to (8.29c). The bottom boundary condition (8.29a) is already in linear form, as is the mass conservation equation (8.28a).

Here is the plan: We shall solve the equation (8.28a) subject to the bottom boundary condition (8.29a) and the top boundary conditions (8.31) and (8.32). We call this the *linear problem* (LP) for ϕ and η. As we shall see, LP completely determines $\phi(x,y,z,t)$ and $\eta(x,y,t)$. Then if we want to know $p(x,y,z,t)$, we can simply calculate it from (8.28b). Of course, our solution is valid only in the limit of infinitesimally small waves, because that was the assumption made in replacing the exact surface boundary conditions by their linear approximations (8.31) and (8.32).

Before tackling LP, we pause to admire the wonderful advantages of linear equations. Suppose that you have found a solution to LP. Let your particular solution be $\phi_1(x,y,z,t)$ and $\eta_1(x,y,t)$. It is actually a whole family of solutions. For, as you can easily verify, if $\phi_1(x,y,z,t)$ and $\eta_1(x,y,t)$ satisfy LP, then so does $C\phi_1(x,y,z,t)$ and $C\eta_1(x,y,t)$, where C is any constant. In other words, every solution has an arbitrary amplitude C, provided only that C is small enough to justify the approximations underlying LP.

But there is more. Suppose you find the solution $\phi_1(x,y,z,t)$ and $\eta_1(x,y,t)$, and your friend finds a completely different solution $\phi_2(x,y,z,t)$ and $\eta_2(x,y,t)$. Then, as you can easily verify, the sum of the two solutions, $\phi_1(x,y,z,t)+\phi_2(x,y,z,t)$ and $\eta_1(x,y,t)+\eta_2(x,y,t)$, also solves LP. This is Postulate #2 writ large. It is called the *superposition principle*, and it applies to any set of linear equations. Solutions can be added together to produce new solutions. This is not true of the general, nonlinear fluid equations.

Now for the solution of LP. First we eliminate η between (8.31) and (8.32) to get a top boundary condition,

$$\frac{\partial^2\phi}{\partial t^2}(x,y,0,t)+g\frac{\partial\phi}{\partial z}(x,y,0,t)=0 \tag{8.33}$$

that involves only ϕ. The equation (8.28a) and the boundary conditions (8.29a) and (8.33) will determine $\phi(x,y,z,t)$. Because of the superposition principle, it is sufficient to look for solutions of the form,

$$\phi(x,y,z,t)=F(z)\sin(kx+ly-\omega t) \tag{8.34}$$

where k and l are arbitrary constants, $F(z)$ is a function to be determined, and ω is a constant to be determined. Substituting (8.34) into (8.28a) yields

$$\frac{d^2F}{dz^2}=\kappa^2F \tag{8.35}$$

where $\kappa^2=k^2+l^2$. The general solution of (8.35) is

$$F(z)={C_1}'e^{\kappa z}+{C_2}'e^{-\kappa z} \tag{8.36}$$

where ${C_1}'$ and ${C_2}'$ are arbitrary constants. However, it will prove more convenient to write this in the equivalent form

$$F(z)=C_1\cosh\left(\kappa(z+H_0)\right)+C_2\sinh\left(\kappa(z+H_0)\right) \tag{8.37}$$

where C_1, C_2 are a different set of arbitrary constants, related to ${C_1}'$, ${C_2}'$ in the obvious way. Substituting

$$\phi(x,y,z,t)=\left[C_1\cosh\left(\kappa(z+H_0)\right)+C_2\sinh\left(\kappa(z+H_0)\right)\right]\sin(kx+ly-\omega t) \tag{8.38}$$

into the bottom boundary condition (8.29a), we obtain

$$(C_1\kappa\sinh(0)+C_2\kappa\cosh(0))\sin(kx+ly-\omega t)=C_2\kappa\sin(kx+ly-\omega t)=0 \tag{8.39}$$

Since this must hold for all (x, y, t), we must have $C_2 = 0$. Thus (8.38) becomes

$$\phi(x, y, z, t) = C_1 \cosh\left(\kappa(z + H_0)\right) \sin(kx + ly - \omega t) \tag{8.40}$$

Substituting (8.40) into the top boundary condition (8.33) yields

$$-\omega^2 C_1 \cosh(\kappa H_0) + g\kappa C_1 \sinh(\kappa H_0) = 0 \tag{8.41}$$

We can satisfy (8.41) by setting $C_1 = 0$, but that would correspond to the trivial solution $\phi \equiv 0$. If on the other hand $C_1 \neq 0$, then (8.41) implies

$$\omega^2 = g\kappa \tanh(\kappa H_0) \tag{8.42}$$

which is the general dispersion relation for water waves! From (8.32) and (8.40) we have

$$\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t}(x, y, 0, t) = \frac{\omega C_1}{g} \cosh(\kappa H_0) \cos(kx + ly - \omega t) \tag{8.43}$$

while from (8.14) and (8.40) we have

$$u = \frac{\partial \phi}{\partial x}(x, y, z, t) = kC_1 \cosh\left(\kappa(z + H_0)\right) \cos(kx + ly - \omega t) \tag{8.44}$$

and

$$w = \frac{\partial \phi}{\partial z}(x, y, z, t) = \kappa C_1 \sinh\left(\kappa(z + H_0)\right) \sin(kx + ly - \omega t) \tag{8.45}$$

Setting

$$A = \frac{\omega C_1}{g} \cosh(\kappa H_0) = \frac{\kappa C_1}{\omega} \sinh(kH_0) \tag{8.46}$$

these become

$$\eta = A \cos(kx + ly - \omega t), \tag{8.47}$$

$$u = A\omega \frac{k}{\kappa} \frac{\cosh\left(\kappa(z + H_0)\right)}{\sinh(\kappa H_0)} \cos(kx + ly - \omega t), \tag{8.48}$$

and

$$w = A\omega \frac{\sinh\left(\kappa(z + H_0)\right)}{\sinh(\kappa H_0)} \sin(kx + ly - \omega t), \tag{8.49}$$

which are just the equations given by Postulate #1. This justifies our use of Postulates #1 and #2, and virtually everything we have done in the previous chapters.

Now let's return to the issue of potential flow. What really justifies (8.14)? The answer to this question goes right to the heart of fluid mechanics. Consider the curl of the velocity field $\mathbf{v}(x, y, z, t)$. It is called the *vorticity*, and it is defined by

$$\mathbf{Q} \equiv \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (8.50)$$

Physically speaking, the vorticity $\mathbf{Q}$ turns out to be twice the angular velocity of the fluid in a reference frame moving with the local fluid velocity. Comparing (8.15) and (8.50), we see that the special conditions satisfied by potential flow are equivalent to the vanishing of the vorticity $\mathbf{Q}$. Conversely, every flow whose vorticity vanishes is a potential flow.

Why should the vorticity vanish? To answer that, we must derive an evolution equation for $\mathbf{Q}$ analogous to the evolution equation (8.10) for the velocity $\mathbf{v}$. We do this by applying the curl operator $(\nabla \times)$ to (8.10). The right-hand side of (8.10) is annihilated, because the curl of a gradient always vanishes. The final result, after a few vector identities, is

$$\frac{\partial \mathbf{Q}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{v} = 0 \quad (8.51)$$

This is an important equation, and a proper understanding of it demands some effort. That would take us much deeper into fluid mechanics than our time allows. Here we only want to make two points. First, waves generated by atmospheric pressure fluctuations do not *acquire* vorticity, because, as we have seen, the curl of the pressure gradient vanishes. Second, according to (8.51), if $\mathbf{Q}$ vanishes initially, then it vanishes for all time. These two points justify (8.14), but all of this is far from obvious, and, historically speaking, it took a very long time to be fully understood.

The interesting thing about $\mathbf{Q}$ is that it is a prerequisite for fluid turbulence. Potential flows are never turbulent. However, (8.51) allows $\mathbf{Q}$ to grow rapidly even if only a small amount of $\mathbf{Q}$ is initially present. *Breaking* waves generate large amounts of $\mathbf{Q}$ and rapidly become turbulent. Thus the theory of potential flow is useful in describing ocean waves up to the point of breaking. Between the breakers and the beach, vorticity must be taken into account.

Assuming that the flow is a potential flow, we have linearized the equations of ideal hydrodynamics to recover the contents of Postulates #1 and #2. However, the general nonlinear equations describe waves with realistically large amplitude. How can we learn about these? One way is to postulate a series expansion of all the dependent variables in powers of the wave amplitude A. The first terms in each series represent linear theory. Higher order terms represent corrections to linear theory. This exercise is very labor intensive! We give only a few results. For a deep water wave moving in the positive x-direction, the expansion procedure yields the *Stokes wave*

$$\text{DW} \qquad \eta = A\cos\theta + \frac{1}{2}kA^2\cos 2\theta + \frac{3}{8}k^2A^3\cos 3\theta + \cdots \tag{8.52}$$

where $\theta = kx - \omega t$. The surface elevation corresponding to (8.52) has sharper crests and flatter troughs than the pure cosine wave. The corresponding phase speed is given by

$$\text{DW} \qquad c^2 = \frac{g}{k}(1 + k^2A^2 + \cdots) \tag{8.53}$$

At leading order, (8.53) agrees with linear theory, in which the phase speed depends only on wavelength. However, at second order, the phase speed depends on the amplitude of the wave, and we find that large-amplitude waves move slightly faster than small-amplitude waves of the same wavelength. The dependence of phase speed on wave amplitude is a hallmark of nonlinearity.

This chapter could be described as a whirlwind introduction to fluid mechanics. Our goal has been to justify the two postulates underlying all our previous results, and to show how these postulates follow from the basic laws of physics—the conservation of mass and momentum. To *really* understand fluid mechanics, you need to take some specialized courses. Maybe you will!

Chapter 9

The shallow-water equations. Tsunamis

Our study of waves approaching the beach had stopped at the point of wave breaking. At the point of wave breaking, the *linear* theory underlying Postulates #1 and #2 breaks down. In chapter 8, we derived the general *nonlinear* equations of fluid mechanics—equations (8.8) and (8.9). These general equations govern wave breaking and the turbulent flow that results. However, the general equations are very difficult to handle mathematically.

In this chapter, we derive simpler but less general equations that apply only to flow in shallow water but still include nonlinear effects. These *shallow-water equations* apply to flow in the surf zone, as well as to tides and tsunamis. The shallow-water equations require that the horizontal scale of the flow be much larger than the fluid depth. In chapter 1 we considered this limit in the context of Postulate #1. There it corresponded to the limit $kH \to 0$, and it led to nondispersive waves traveling at the speed $c = \sqrt{gH}$. We will see these waves again.

We start with the general equations for three-dimensional fluid motion derived in chapter 8—equations (8.8) and (8.9). Now, however, we specifically avoid the assumption (8.14) of potential flow. The assumption of potential flow is invalid for breaking waves because wave-breaking generates vorticity, and it is invalid for tides because the Earth's rotation is a source of vorticity in tides.

We start by writing out (8.8) and (8.9) in full detail. The mass conservation equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{9.1}$$

and the equations for momentum conservation in each direction are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} \tag{9.2a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} \tag{9.2b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - g \tag{9.2c}$$

The shallow-water approximation is based upon two assumptions. The first assumption is that the left-hand side of (9.2c)—the vertical component, Dw/Dt, of the acceleration—can be neglected. If the vertical acceleration is negligible, then (9.2c) becomes

$$\frac{\partial p}{\partial z} = -g \tag{9.3}$$

which is often called the *hydrostatic equation*. We get an expression for the pressure by integrating (9.3) with respect to z, from the arbitrary location (x, y, z) to the point $(x, y, \eta(x, y))$ on the free surface directly above. (Since this is being done at a fixed time, we suppress the time argument.) The integration yields

$$0 - p(x, y, z) = -g\left(\eta(x, y) - z\right) \tag{9.4}$$

because the pressure vanishes on the free surface. Thus, if the vertical acceleration is negligible, the pressure

$$p(x, y, z, t) = g\left(\eta(x, y, t) - z\right) \tag{9.5}$$

is determined solely by the weight of the overlying water. Substituting (9.5) back into (9.2a) and (9.2b), we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -g\frac{\partial \eta}{\partial x} \tag{9.6a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -g\frac{\partial \eta}{\partial y} \tag{9.6b}$$

Next we apply the second of the two assumptions of shallow-water theory. We assume that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \tag{9.7}$$

That is, we assume that the horizontal velocity components are independent of z. The fluid motion is said to be *columnar*. As with the first assumption, it remains to be seen that (9.7) is an appropriate assumption for shallow-water flow. However, (9.7) seems reasonable, because, as we saw in chapter 1, SW waves have a z-independent horizontal velocity; see (1.19c).

If (9.7) holds, then (9.6) take the simpler forms

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} \tag{9.8a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \eta}{\partial y} \tag{9.8b}$$

The two horizontal momentum equations (9.8) represent two equations in the three dependent variables $u(x, y, t)$, $v(x, y, t)$, and $\eta(x, y, t)$. To close the problem, we need a third equation in these same variables.

The third equation comes from the mass conservation equation (9.1). To get it, we integrate (9.1) from the rigid bottom at $z = -H(x, y)$ to the free surface at $z = \eta(x, y, t)$ and apply the kinematic boundary conditions. Since u and v are independent of z, the integration yields

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w(x, y, \eta) - w(x, y, -H) = 0 \tag{9.9}$$

where

$$h(x, y, t) \equiv \eta(x, y, t) + H(x, y) \tag{9.10}$$

is the vertical thickness of the water column. See figure 9.1.

The kinematic boundary conditions state that fluid particles on the boundaries remain on the boundaries. Thus at the free surface we have

$$0 = \frac{D}{Dt}\left(z - \eta(x, y, t)\right) = w - \frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y} \tag{9.11}$$

and at the rigid bottom we have

$$0 = \frac{D}{Dt}\left(z + H(x, y)\right) = w + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} \tag{9.12}$$

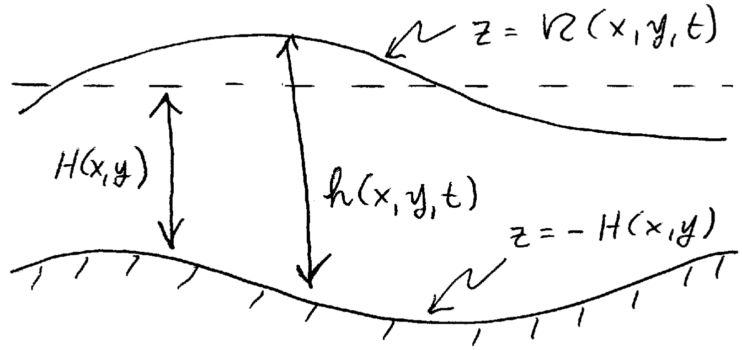

Figure 9.1: The vertical thickness of the fluid is $h = \eta + H$.

Using (9.11) and (9.12) to eliminate the vertical velocities in (9.9), we obtain

$$h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=0 \tag{9.13}$$

Combining terms in (9.13) we have

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)+\frac{\partial}{\partial y}(hv)=0 \tag{9.14}$$

which can also be written in the form

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left(u(\eta+H)\right)+\frac{\partial}{\partial y}\left(v(\eta+H)\right)=0 \tag{9.15}$$

Equation (9.15) is the required third equation in the variables $u(x,y,t)$, $v(x,y,t)$, and $\eta(x,y,t)$. However, the definition (9.10) allows us to use either $\eta(x,y,t)$ or $h(x,y,t)$ as the third dependent variable.

Although their derivation has taken a bit of work, the shallow-water equations (9.8,9.15) make good physical sense all on their own. Take the mass conservation equation in the form (9.14). In vector notation it is

$$\frac{\partial h}{\partial t}+\nabla\cdot(h\mathbf{u})=0 \tag{9.16}$$

where $\mathbf{u}=(u,v)$ is the z-independent horizontal velocity. Equation (9.14) or (9.16) is a conservation law of the general form (8.1). According to (9.16), the fluid thickness h increases if the mass flux $h\mathbf{u} = (hu, hv)$ converges. The momentum equations (9.8) can be written in the vector form

$$\frac{D\mathbf{u}}{Dt} = -g\nabla\eta \tag{9.17}$$

where now

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u(x,y,t)\frac{\partial}{\partial x} + v(x,y,t)\frac{\partial}{\partial y} \tag{9.18}$$

is the time derivative following a moving fluid column. According to (9.17), fluid columns are accelerated *away* from where the sea surface elevation η is greatest. In other words, gravity tends to flatten the free surface.

Before saying anything more about the general, nonlinear form—(9.8) and (9.15)—of the shallow water equations, we consider the corresponding linear equations. Suppose, as in chapter 8, that the motion is a slight departure from the state of rest, in which $u = v = \eta = 0$. Then we may neglect the products of u, v and η in (9.8) and (9.15). The resulting equations are the *linear* shallow-water equations:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{9.19a}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \tag{9.19b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = 0 \tag{9.19c}$$

The linear equations (9.19) may be combined to give a single equation in a single unknown. Taking the time derivative of (9.19c) and substituting from (9.19a) and (9.19b), we obtain

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial x}\left(gH\frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(gH\frac{\partial \eta}{\partial y}\right) \tag{9.20}$$

If we specialize (9.20) to the case of one space dimension, we have

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial x}\left(gH\frac{\partial \eta}{\partial x}\right) \tag{9.21}$$

and if we further specialize (9.21) to the case of constant $H = H_0$ (corresponding to a flat bottom), we have

$$\frac{\partial^2 \eta}{\partial t^2} = c^2\frac{\partial^2 \eta}{\partial x^2} \tag{9.22}$$

where $c^2 = gH_0$ is a constant.

The equation (9.22) is often called the 'wave equation' despite the fact that many, many other equations also have wave solutions. However, (9.22) does have the following remarkable property: The general solution of (9.22) is given by

$$\eta(x,t) = F(x - ct) + G(x + ct) \tag{9.23}$$

where F and G are arbitrary functions. In other words, the solution of (9.22) consists of an arbitrary shape translating to the right at constant speed c and another arbitrary shape translating to the left at the same speed. In the remainder of this chapter, we use the *linear* shallow-water equations as the basis for a discussion of tsunamis. In chapter 10 we use the *nonlinear* shallow-water equations to say something about wave breaking.

Most tsunamis are generated by earthquakes beneath the ocean floor. (Volcanic eruptions and submarine landslides also generate tsunamis.) The earthquakes are associated with the motion of the Earth's tectonic plates. Most earthquakes occur at plate boundaries when the energy stored in crustal deformation is released by a sudden slippage. Most tsunamis occur in the Pacific Ocean, which sees 3 or 4 major tsunamis per century. Tsunamis are especially likely to occur when a section of the ocean bottom is thrust vertically upward or downward. Because this happens very suddenly, the ocean responds by raising or lowering its surface by about the same amount. If this area of raising or lowering is broader than the ocean is deep, then the subsequent motion is governed by the shallow-water equations. The very tragic tsunami of 11 March, 2011, was generated by a magnitude 9.0 earthquake on a thrust fault in the subduction zone just east of the Japanese island of Honshu. The sea floor there rose suddenly some 5 to 8 meters along a 300-mile-long rupture zone that was only about 40 miles offshore.

In deep water tsunamis are well described by the linear shallow-water equation (9.20). (Most tsunami models also include the Coriolis force resulting from the Earth's rotation, but this is of secondary importance.) To understand the physics, we consider an idealized, one-dimensional example. Imagine that an infinitely long section of seafloor, with width W in the x-direction, suddenly experiences a vertical drop of distance d. This drop is quickly transmitted to the ocean surface. Assuming that W is much greater than the ocean depth, and that the seafloor is approximately flat (despite the drop), (9.22) governs the subsequent motion. Since (9.22) has two time derivatives, it requires two initial conditions. One of these is the sea surface

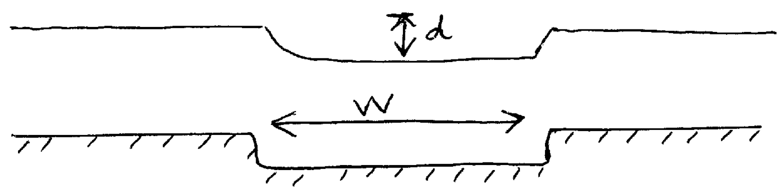

Figure 9.2: The initial surface displacement resulting from a sudden drop in the sea floor.

elevation just after the drop,

$$\eta(x,0) = f(x) \equiv \begin{cases} -d, & |x| < W/2 \\ 0, & |x| > W/2 \end{cases} \tag{9.24}$$

and the other is the initial horizontal velocity. Refer to figure 9.2. For simplicity, we suppose that $u(x,0) = 0$; the horizontal velocity vanishes initially. It follows from this and the one-dimensional form

$$\frac{\partial \eta}{\partial t} = -H_0 \frac{\partial u}{\partial x} \tag{9.25}$$

of (9.19c) that

$$\frac{\partial \eta}{\partial t}(x,0) = 0 \tag{9.26}$$

Thus the problem reduces to choosing the arbitrary functions in (9.23) to satisfy the initial conditions (9.24) and (9.26). Substituting (9.23) into (9.26) we find that

$$-cF'(x) + cG'(x) = 0 \tag{9.27}$$

for all x. Therefore $F(x) = G(x)$, and it then follows from (9.24) that $F(x) = G(x) = \frac{1}{2}f(x)$. Thus the solution to the problem is

$$\eta(x,y) = \frac{1}{2}f(x - ct) + \frac{1}{2}f(x + ct) \tag{9.28}$$

We have already encountered a solution like this in chapter 3; see (3.35). According to (9.28) the sea-surface depression (9.24) splits into two parts which move symmetrically apart with speed c. An upward-thrusting seafloor would produce a local sea-surface elevation that would split apart in the same way. A non-vanishing initial velocity would destroy the directional symmetry.

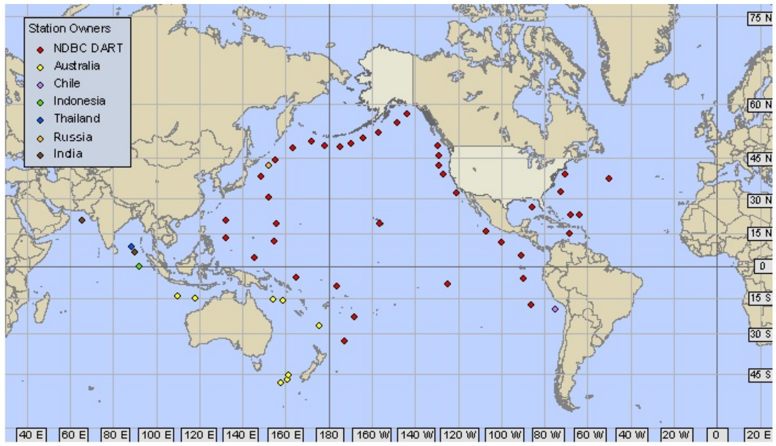

Figure 9.3: Locations of the DART buoys in NOAA's tsunami warning system.

The first thing to say about tsunamis is that they are remarkably fast. The average ocean depth is about 4 km. This corresponds to a wave speed $c = \sqrt{gH_0}$ of about 200 meters per second, or about 1000 km per hour. This doesn't give much warning time. Tsunami warnings are issued as soon as seismometers record a big earthquake. (This happens quickly, because seismic waves travel *very* rapidly through the solid earth.) However, not all big earthquakes generate tsunamis. For example, an earthquake in a region with a flat ocean bottom and in which most of the crustal motion is horizontal, would produce no tsunami. However, even the biggest tsunamis correspond to ocean bottom displacements of only a few meters. Since these relatively small displacements occur on a horizontal scale of many kilometers, tsunamis are hard to detect in the deep ocean. Because of their long wavelengths, they cannot be observed directly from ships. However, tsunamis can be detected by bottom pressure gauges that are acoustically linked to nearby moored DART buoys (an acronym for Deep-ocean Assessment and Reporting of Tsunami). DART buoys (figure 9.3) transmit an estimate of sea-surface height via satellite. This provides the confirmation that a tsunami is on its way. Figure 9.4 shows time series of the sea surface height measured by DART during the 2011 Honshu tsunami. The maximum surface elevation of 1.78 meters was the highest ever recorded by a DART installation. For much more about the Honshu tsunami, including an animated solution of the shallow-water equations, see the NOAA website (http://www.tsunami.noaa.gov).

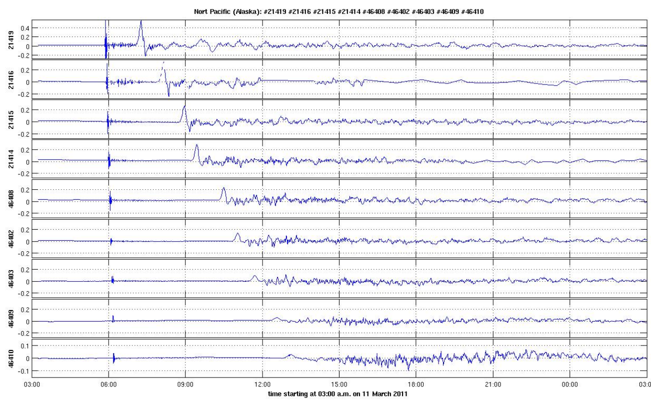

Figure 9.4: The 2011 Honshu tsunami as seen by DART buoys in the northwestern Pacific. Bottom pressure versus time with the tides removed.

The typical tsunami is a short series of sea-surface elevations and depressions —a wavepacket—with wavelengths greater than 100 km and periods in the range 10 minutes to one hour. The leading edge of the packet can correspond to a sea-surface depression or to an elevation, but an initial depression is considered more dangerous because it often entices bathers to explore the seabed exposed by the initially rapidly receding water.

To predict tsunami amplitudes with any reasonable accuracy one must solve (9.20) using the realistic ocean bathymetry $H(x, y)$. This definitely requires the use of a big computer. The initial conditions must be interpolated from the buoy measurements. However, since time is so short, this whole process must be completely automated; such an automatic system is still under development. It is important to emphasize that tsunamis feel the effects of the bathymetry everywhere. In this they are unlike the much shorter wind-generated waves, which feel the bottom only very close to shore. The bathymetry steers and scatters tsunamis from the very moment they are generated.

Tsunamis become dangerous as they enter shallow coastal waters. Energy that is initially spread through a water column 4 km thick becomes concentrated in a few tens of meters. Of course much energy is dissipated by bottom friction and some is reflected away from shore, but the wave amplitude increases rapidly despite these losses. Shoaling waves of elevation form *bores* with heights that sometimes reach 50 to 100 feet at the shoreline. The

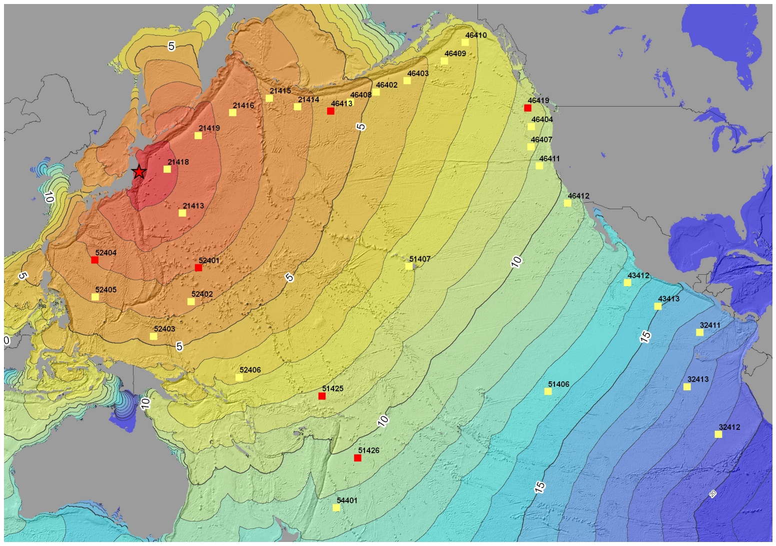

Figure 9.5: Arrival times of the first wave in the 2011 Honshu tsunami, based on eyewitness reports, tide gauges, and DART data. Each contour line represents one hour.

maximum height of the Honshu tsunami was about 120 feet.

We have previously considered shoaling wind-generated waves using the ‘slowly varying’ assumption that the ocean depth changes only slightly over a wavelength. Using the facts that the frequency and the shoreward energy flux are constants, we predicted the amplitude increase in these waves. However, slowly varying theory does not apply accurately to tsunamis, because the wavelengths of tsunamis are so large. On the global scale, the earthquake that generates the tsunami resembles a ‘point source.’ At large distances from the source, it resembles a radially symmetric wavepacket with a typical wavelength of 500 km. Figure 9.5 shows the location of the Honshu tsunami’s first arrival at hourly intervals. Note how the wave slows down in shallow water.

The amplitude of the tsunami decreases as the wave spreads its energy over a circle of increasing radius. In this respect the one-dimensional solution (9.28) is misleading, as was the corresponding one-dimensional solution in chapter 4. However, the corresponding two-dimensional problem is much more difficult; even in the linear case it requires mathematical methods that you have probably just begun to learn. For that reason, and because the *nonlinear* shallow-water equations are easily treatable *only* in one space dimension, we stick to one-dimensional examples.

It is interesting to examine a case in which shallow-water waves interact with bathymetry that varies rapidly on the scale of the wave. Although this violates the assumptions on which the equations were derived, a more detailed analysis shows that the solutions are still surprisingly accurate. We therefore consider the case of a shallow-water wave propagating from $x = -\infty$ toward a step at $x = 0$ where the water depth changes from the constant value H_1 to the constant value H_2 (figure 9.6). The incoming wave has the form $\eta = f(x - c_1 t)$, where the function $f(s)$ (with s a dummy argument) is completely arbitrary. This 'wave' could be a wavetrain such as $f(s) = \cos(k_1 s)$, or it could be a pulse or wavepacket better resembling a tsunami. The step could represent the edge of the continental shelf (in the case of the tsunami) or a submerged breakwater or bar (in the case of a surf-zone wave). The solution must take the form

$$\eta(x,t) = \begin{cases} f(x - c_1 t) + G(x + c_1 t), & x < 0 \\ F(x - c_2 t), & x > 0 \end{cases} \tag{9.29}$$

where $c_1 = \sqrt{gH_1}$ and $c_2 = \sqrt{gH_2}$. Here, G represents the reflected wave and F represents the transmitted wave. The situation is that f is a given function, but the functions G and F remain to be determined.

We find G and F by matching the solution across the step. The matching conditions are that the pressure and the mass flux be continuous at the step. Continuity of pressure implies continuity of the surface elevation η. Thus

$$\eta(0^-, t) = \eta(0^+, t) \tag{9.30}$$

or, using (9.26),

$$f(-c_1 t) + G(c_1 t) = F(-c_2 t) \tag{9.31}$$

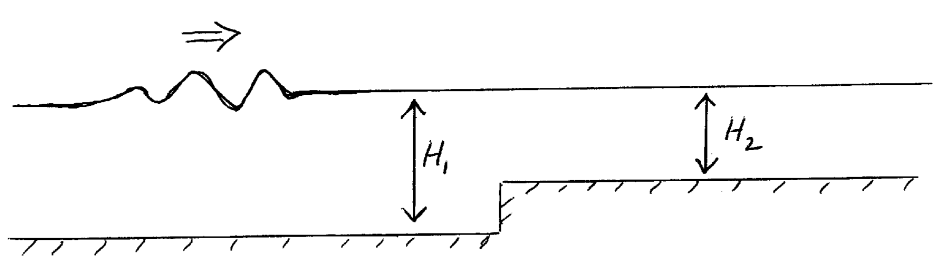

Figure 9.6: Wave impinging on a step.

which must hold for all time. Continuity of mass flux implies

$$H_1 u(0^-, t) = H_2 u(0^+, t) \tag{9.32}$$

To express the condition (9.32) in terms of η, we take its time derivative and substitute from (9.19a) to obtain

$$-gH_1 \frac{\partial \eta}{\partial x}(0^-, t) = -gH_2 \frac{\partial \eta}{\partial x}(0^+, t) \tag{9.33}$$

Substituting (9.29) into (9.33) we obtain

$$-gH_1 \left(f'(-c_1 t) + G'(+c_1 t) \right) = -gH_2 F'(-c_2 t) \tag{9.34}$$

The matching conditions (9.31) and (9.34) determine G and F in terms of the given function f. Let

$$\alpha = \sqrt{\frac{H_2}{H_1}} \tag{9.35}$$

Then $c_2 = \alpha c_1$, and (9.31) may be written

$$f(s) + G(-s) = F(\alpha s) \tag{9.36}$$

while (9.34) may be written

$$f'(s) + G'(-s) = \alpha^2 F'(\alpha s) \tag{9.37}$$

Here s is simply a dummy variable. The integral of (9.37) is

$$f(s) - G(-s) = \alpha F(\alpha s) + C \tag{9.38}$$

where C is a constant of integration. Since this constant merely adds a constant value to η on each side of the step, we set $C = 0$. Then, solving (9.36) and (9.38) for G and F, we obtain

$$F(s) = \frac{2}{1+\alpha} f(s/\alpha) \tag{9.39}$$

and

$$G(s) = \frac{1-\alpha}{1+\alpha} f(-s) \tag{9.40}$$

Thus the complete solution is

$$\eta(x,t) = \begin{cases} f(x - c_1 t) + \frac{1-\alpha}{1+\alpha} f(-x - c_1 t), & x < 0 \\ \frac{2}{1+\alpha} f\left((x - c_2 t)/\alpha\right), & x > 0 \end{cases} \tag{9.41}$$

The solution (9.41) satisfies the wave equation on each side of the step and the matching conditions across the step.

The solution (9.41) is valid for any f you choose. But suppose you choose $f(s) = A\cos(k_1 s)$ corresponding to an incoming basic wave. We leave it as an exercise for you to show that, for this particular choice of f, the solution (9.41) takes the form

$$\eta(x,t) = \begin{cases} A\cos(k_1 x - \omega t) + \frac{1-\alpha}{1+\alpha} A\cos(k_1 x + \omega t), & x < 0 \\ \frac{2}{1+\alpha} A\cos(k_2 x - \omega t), & x > 0 \end{cases} \tag{9.42}$$

where A and k_1 are arbitrary constants, and ω and k_2 are given by

$$\omega = k_1\sqrt{gH_1} = k_2\sqrt{gH_2} \tag{9.43}$$

In both (9.41) and (9.42), the relative amplitudes of the reflected and transmitted waves depend solely on α, which ranges from 0 to ∞. Suppose that f represents an elevation of the sea surface. That is, suppose $f(s)$ is positive near $s = 0$ and vanishes elsewhere. If α is very small—that is, if H_2 is very small—then this pulse of elevation is reflected from the step without a change in size or in sign. As α increases toward 1, the reflected pulse diminishes in size, vanishing when $\alpha = 1$ (no step). At this point, the transmitted pulse is identical to the incoming pulse. As α increases from 1 to ∞ (pulse moving from shallow water to deep water), the size of the reflected pulse increases again, but in this range the reflected pulse has the opposite sign from the incoming pulse. That is, an elevation reflects as a depression.

This is a good place to say something about energy. We leave it to you to show that the linearized shallow-water equations (9.19) imply an energy-conservation equation of the form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}Hu^2 + \frac{1}{2}Hv^2 + \frac{1}{2}g\eta^2\right) + \frac{\partial}{\partial x}\left(gHu\eta\right) + \frac{\partial}{\partial y}\left(gHv\eta\right) = 0 \tag{9.44}$$

The energy per unit horizontal area is

$$\rho\left(\frac{1}{2}Hu^2 + \frac{1}{2}Hv^2 + \frac{1}{2}g\eta^2\right) \tag{9.45}$$

where ρ is the constant mass density. If H is constant, the basic wave

$$\eta = A\cos\left(k(x-\sqrt{gH_0}t)\right) \tag{9.46a}$$

$$u = \frac{A\omega}{kH_0}\cos\left(k(x-\sqrt{gH_0}t)\right) \tag{9.46a}$$

is a solution of the linearized shallow-water equations. (Note that (9.46) agrees with (1.19)). Thus, remembering that the average of cosine squared is one half, the energy per unit horizontal area of the basic wave, averaged over a wavelength or period, is

$$E = \rho\left(\frac{1}{4}\frac{A^2\omega^2}{k^2{H_0}^2} + \frac{1}{4}gA^2\right) = \frac{1}{2}g\rho A^2 \tag{9.47}$$

because $\omega^2 = gH_0k^2$. This finally justifies (2.37), our much used assumption that the wave energy is proportional to the square of the wave amplitude. Using this result and the fact that the energy flux equals E times the group velocity, you should be able to show that, in the solution (9.42), the energy flux of the incoming wave equals the sum of the energy fluxes in the reflected and transmitted waves.

Chapter 10

Breakers, bores and longshore currents

There is lots more to say about linear, shallow-water waves, but now we want to say something about the more general, nonlinear case. To keep the math as simple as possible, we continue to restrict ourselves to one space dimension.

The nonlinear shallow-water equations come into play when the linear equations break down. That happens when the wave amplitudes become too large to justify the small-amplitude assumption behind linear theory. In one dimension, the nonlinear shallow-water equations (9.8) and (9.14) may be written in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial (h-H)}{\partial x} \tag{10.1a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \tag{10.1b}$$

In the linear (i.e. small-amplitude) limit, (10.1) take the forms

$$\frac{\partial u}{\partial t} = -g\frac{\partial (h-H)}{\partial x} \tag{10.2a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0 \tag{10.2b}$$

where $H(x)$ is a given function. Equations (10.2) are equivalent to the one-dimensional version of (9.19). The linear equations (10.2) apply to long waves *seaward* of the breaker zone. In the breaker zone itself we must use the more exact equations (10.1).

The nonlinear property of (10.1) makes them very hard to solve, but we make progress if we are willing to regard the $H(x)$ in (10.1a) as a constant. Since the increase in wave amplitude is associated with a decrease in $H(x)$, such an approximation seems very hard to justify. However, we take the following viewpoint: The decreasing water depth causes the wave amplitude to increase to the point where nonlinear effects become important. These nonlinear effects then cause a *rapid* steepening of the waves. This steepening occurs before the depth can decrease much further. Therefore we are justified in considering (10.1) without the dH/dx-term, namely

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} \tag{10.3a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \tag{10.3b}$$

If you find this sort of reasoning unconvincing, you might be comforted to know that (10.1) have in fact been solved; we refer to that solution below. However the solution of (10.1) involves some mathematical tricks that are a bit too advanced for this course. The solution of (10.3) is much easier, yet it still tells us something about wave steepening and breaking.

It is a somewhat astonishing fact that the two equations (10.3) can be written in the forms

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)(u+2c) = 0 \tag{10.4a}$$

$$\left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)(u-2c) = 0 \tag{10.4b}$$

where now we define $c(x,t) \equiv \sqrt{gh(x,t)}$. In (10.4) we regard $u(x,t)$ and $c(x,t)$ as the dependent variables. To show that (10.4) are equivalent to (10.3), simply take the sum and difference of (10.4).

According to (10.4), an observer moving at the speed $u+c$ always sees the same value of $u+2c$, while an observer moving at the speed $u-c$ always sees the same value of $u-2c$. An easy way to satisfy (10.4b) would be to set $u-2c$ equal to a constant. Why would we want to do that? Suppose we have a wave moving to the right—toward positive x—and into still water. If $u < c$, the observer corresponding to (10.4b) is moving to the left. He sees the same value of $u - 2c$ that he saw at $x = +\infty$, where the water is at rest. Hence

$$u - 2c = -2\sqrt{gH_0} \equiv -2c_0 \tag{10.5}$$

everywhere in the flow, where H_0 is the constant value of h in the quiescent region at $x = +\infty$. Substituting (10.5) into (10.3b) yields

$$\frac{\partial h}{\partial t} + C(h)\frac{\partial h}{\partial x} = 0 \tag{10.6a}$$

where

$$C(h) \equiv 3\sqrt{gh} - 2\sqrt{gH_0} \tag{10.6b}$$

Equation (10.6a) is an equation in h alone.

According to (10.6a), an observer moving to the right at speed $C(h)$ always sees the same value of h. Suppose that the initial condition $h(x, 0) \equiv h_0(x)$ is given. The observer initially at x_1 moves along the straight line

$$x = x_1 + C(h_0(x_1))t \tag{10.7}$$

and always observes the value $h = h_0(x_1)$. The line (10.7) is called a *characteristic*. Every point on the x-axis corresponds to a characteristic. Each characteristic is defined by its x-intercept x_1. Defining

$$C(x_1) \equiv C(h_0(x_1)) = 3\sqrt{gh_0(x_1)} - 2\sqrt{gH_0} \tag{10.8}$$

we see that each characteristic has a constant speed $dx/dt = C(x_1)$. For typical initial conditions—$h_0(x_1)$ not extremely different from H_0—all the speeds are positive. However if $C'(x_1) < 0$ then the speeds of the characteristics decrease to the right, and the characteristics must eventually cross. Since $C(x_1)$ decreases wherever $h_0(x_1)$ decreases, characteristics eventually cross if the initial depth satisfies $dh_0/dx < 0$ anywhere within the domain. When the characteristics cross, h becomes triple-valued, and the solution breaks down. The crossing point corresponds to an infinite surface slope, and hence to wave breaking.

When and where does this first occur? To answer that, we must find the first time at which two observers arrive at the same point with different values of h. Let the first observer follow (10.7), always observing $h = h_0(x_1)$. Let the second observer follow

$$x = x_2 + C(h_0(x_2))t \tag{10.9}$$

always observing $h = h_0(x_2)$. At the crossing point,

$$x_1 + C(x_1)t = x_2 + C(x_2)t \tag{10.10}$$

The first crossing of characteristics must occur when $x_2 = x_1 + \delta x$, where δx is infinitesimal. Then (10.10) becomes

$$x_1 + C(x_1)t = x_1 + \delta x + C(x_1)t + C'(x_1)\delta x\, t \tag{10.11}$$

Canceling terms, we find that wave breaking first occurs when

$$t = \frac{-1}{C'(x_1)} \tag{10.12}$$

That is, breaking occurs at the time (10.12) on the characteristic corresponding to the most negative value of ${h_0}'(x)$. This corresponds to the point at which the initial surface slope decreases most rapidly to the right.

As a model of wave breaking, this calculation leaves much to be desired. For one thing, it predicts that wave breaking *always* eventually occurs, no matter how small the initial wave amplitude. However, observations and calculations using more accurate approximations show that some waves *never* break. Nevertheless, there is one feature of the solution to (10.3) that is informative and certainly correct: When waves do break, it is because their deeper parts—the crests—move faster that their shallow parts—the troughs—and hence overtake them.

In 1958 George Carrier and Harvey Greenspan produced a stunning, exact, analytical solution of (10.1) for a uniformly sloping beach—a beach with a constant value of $s = |dH/dx|$. Figure 10.1 shows their solution as a dashed line, at three times in the wave cycle. In the case shown, the beach slope s is sufficiently large that the incoming wave (from the right in this case) does not break, but instead reflects to form a standing wave. (The solid line in the figure corresponds to a numerical solution of the same problem, and the comparison was made to test the numerical algorithm.)

The Carrier-Greenspan solution breaks down, implying that the incoming wave breaks (instead of reflecting) if

$$s^2 < \frac{\omega^2 A_{shore}}{g} \tag{10.13}$$

where A_{shore} is the wave amplitude at the shore, that is, half the vertical distance between high water and low water on the beach. For waves with

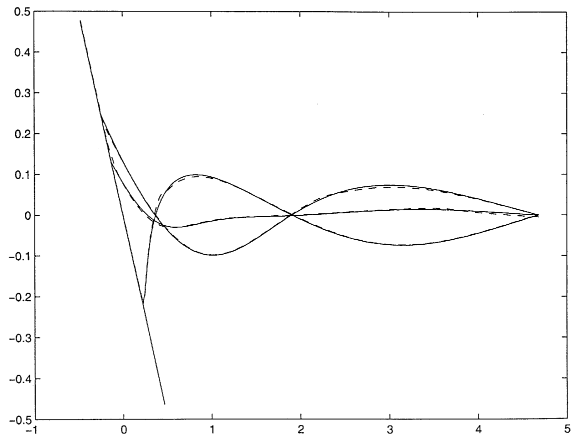

Figure 10.1: Carrier-Greenspan solution of a standing wave on a uniformly sloping beach.

$A_{shore} = 1$m and 10-second period (10.13) predicts wave-breaking on beaches with slopes less than 0.20. Since very few beaches are steeper than this, wave breaking is the rule. Figure 10.1 depicts the Carrier-Greenspan solution near the limit of wave breaking; the vertical scale has been exaggerated for ease of viewing.

One problem with the Carrier-Greenspan solution is that it simultaneously assumes shallow-water dynamics and a constant bottom slope. These two assumptions are incompatible! In regions for which the water depth is much greater than the wavelength, shallow-water dynamics does not apply. In 1963 Joseph Keller matched the Carrier-Greenspan solution to the solution of the linear-waves equations in deep water—more complicated mathematics! Keller's analysis provides the link between the wave amplitude A_∞ in deep water and the amplitude A_{shore} at the shoreline. Keller found that

$$A_{shore} = \sqrt{\frac{2\pi}{s}} A_\infty \tag{10.14}$$

for waves that reach the shore without breaking. Substituting (10.14) into (10.13), we obtain the criterion for wave breaking,

$$s^{5/2} < \sqrt{2\pi} k_\infty A_\infty \tag{10.15}$$

in terms of the wave amplitude A_∞ and wavenumber k_∞ in deep water. The

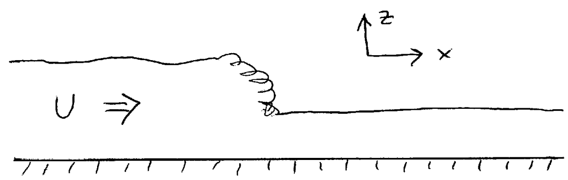

Figure 10.2: A turbulent bore.

breaking criterion (10.15) is equivalent to

$$\frac{s}{(k_\infty A_\infty)^{2/5}} < (2\pi)^{1/5} = 1.44 \tag{10.16}$$

Observations suggest that the breaking of waves on beaches is governed by the 'surf similarity parameter'

$$\chi \equiv \frac{s}{\sqrt{k_\infty A_\infty}} \tag{10.17}$$

which is proportional to the ratio of the bottom slope to the wave slope in deep water. When χ is greater than about 1.9, wave breaking does not occur, and the waves reflect from the beach. When is about 1.9, the waves break right at the water line. When $0.25 < \chi < 1.9$ the waves break offshore as 'plunging breakers': the front face steepens and overturns to form a jet that plunges into the water ahead of the wave. When $\chi < 0.25$ the waves break as 'spilling breakers': the wave retains its symmetrical appearance, but white (aerated) water appears at the crest and subsequently spreads down over the front face of the wave.

What happens *after* waves break? Frequently they form turbulent bores, nearly discontinuous changes in surface height that resemble moving steps (figure 10.2). The turbulence in these bores dissipates energy very rapidly; the bores rapidly diminish in height. However, apart from the momentum transferred to the ocean bottom through bottom drag, the momentum of the fluid is conserved. As the waves dissipate, this momentum is converted to the momentum of currents.

Dissipating waves drive currents in the direction of their wavevector $\mathbf{k}$. Thus waves normally incident on a uniformly sloping beach drive currents directly toward the shore. This leads to *set up*—an increase of surface elevation at the shoreline (figure 10.3). The pressure gradient associated with

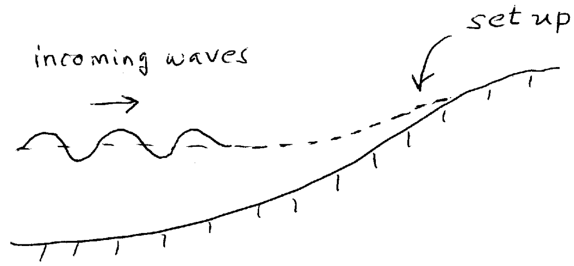

Figure 10.3: Incoming waves lead to *set up* and an opposing pressure gradient.

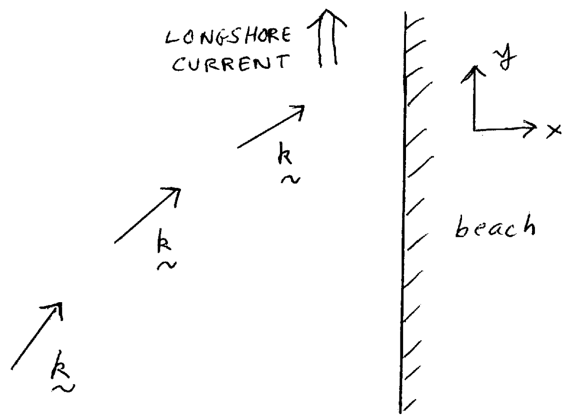

Figure 10.4: When the wavevector **k** has a longshore component, wave breaking drives a longshore current.

set up drives an *offshore* current that cancels the shoreward drift of fluid particles associated with the incoming waves.

Even on a uniformly sloping beach, things are seldom so simple. The incoming waves are never exactly normal to the shoreline. Although refraction turns the wavevector **k** toward the beach, **k** retains a longshore component. When the wave breaks, it drives a longshore current in the direction of its **k** (figure 10.4).

A deeper analysis of this situation shows that a wavetrain with average energy E creates a current with momentum

$$\mathbf{U} = \frac{E}{c}\frac{\mathbf{k}}{|\mathbf{k}|} \tag{10.18}$$

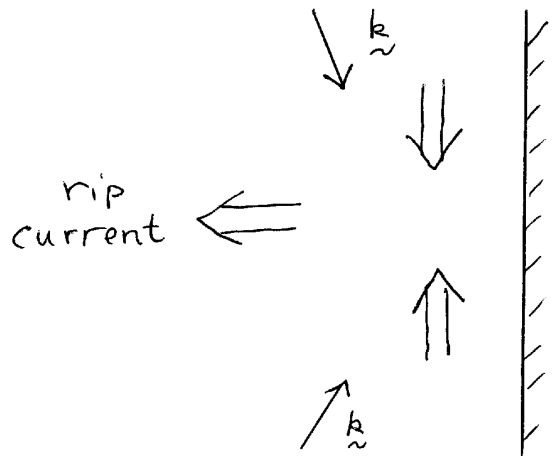

Figure 10.5: Longshore currents converge to form a rip current.

where **k** is the wavevector of the wave and c is its phase velocity. The right-hand side of (10.18) is called the *pseudomomentum* of the wave. If E has units of velocity squared, then **U** is the velocity of the current that results from wave-breaking. For the reasons stated above, it is the longshore component of (10.18) that is most important.

On non-uniformly sloping beaches such as Scripps beach, refraction can turn wavevectors *away* from the shoreline. For example, refraction by the two submarine canyons bends **k** towards Scripps pier from both north and south. The breaking of the northward propagating wave drives a northward flowing longshore current, and conversely. The convergence of these two longshore currents almost certainly explains the rip current that is almost always present about 300 meters south of Scripps pier (figure 10.5).

The currents created by wave breaking in the surface zone are never steady. Even in the case of the uniformly sloping beach, the longshore current may become unstable, breaking up into eddies. It seems best to regard flow in the surf zone as consisting of two fields: a wave field that drives currents as the waves break, and a current field that—along with depth variations—refracts the incoming waves. This course has offered a fairly complete description of the wave field up to the point where wave breaking transforms the waves into currents. What is the corresponding description of the current field?

We cannot go into the details, but the currents are governed by the non-linear shallow-water equations. Because the timescale of the currents is much

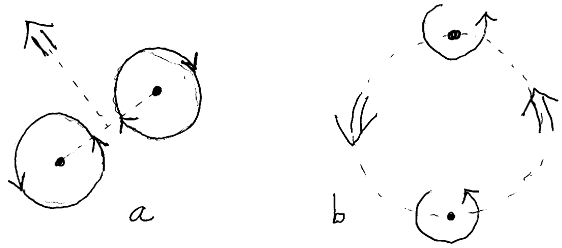

Figure 10.6: (a) Equal-strength counter-rotating vortices move in a direction perpendicular to a line joining their centers, while (b) like-signed vortices move in a circular trajectory.

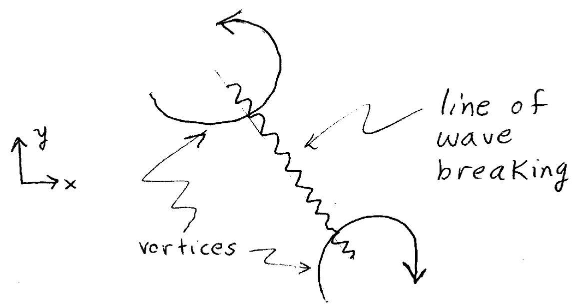

Figure 10.7: A breaking wave generates vortices at its edges.

longer than the period of the waves, the currents see the ocean surface as a rigid lid. This simplifies the problem enormously; the currents then obey the dynamics of what is called “two-dimensional turbulence.” A useful idealization of two-dimensional turbulence regards the flow as consisting of a large number of point vortices. Each vortex induces a circular flow about its center. The direction of the circular flow may be either clockwise or counterclockwise, depending on the sign of the vortex. Each vortex is swept along in the flow caused by all the other vortices. Thus two vortices with equal strengths and opposite signs move on a straight line perpendicular to the line between their centers (figure 10.6a), while two vortices of the same sign move in a circle (figure 10.6b). A breaking wave crest generates vortices at its edges, as shown in figure 10.7. These vortices propel each other toward the shore, where they encounter image vortices that enforce the boundary condition of no flow into the shoreline. These image vortices cause the real vortices to move apart (figure 10.8). As vortices move parallel to the shoreline, they

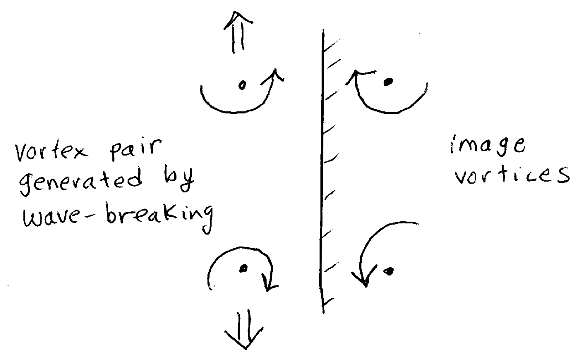

Figure 10.8: The vortex pair separates as it approaches the shoreline.

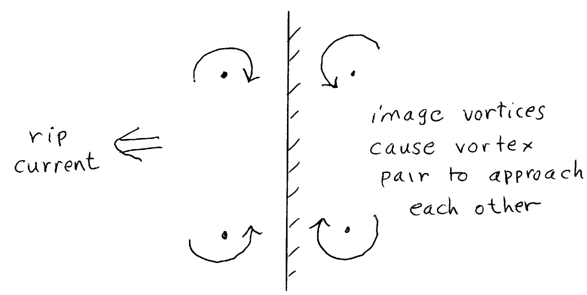

Figure 10.9: Converging vortices induce an offshore current.

encounter other vortices, moving in the opposite direction. Converging vortices produce an offshore current (figure 10.9). In reality, this is a turbulent process, involving many vortices. The figures show what happens in very idealized cases.

There is one further feature of surf zone waves that should be mentioned. The tendency for incoming waves to appear in sets means that the set up at the shore varies on a timescale of minutes. This variation generates low-frequency waves that propagate back toward the deep ocean. Although these *infragravity waves* cannot be directly observed, they are clearly visible in wave spectra. If the direction of the outgoing infragravity waves is not perfectly normal to the shoreline, then the outgoing infragravity waves are refracted back toward shore, forming low-frequency *edge waves* that are trapped along the shore. The edge waves behave like traveling waves in the longshore direction, and like standing waves in the cross-shore direction.

None of these post-breaking phenomena can be adequately explained without the heavy use of fluid mechanics. The foregoing discussion merely conveys the flavor of the subject. Even the full theory seems inadequate to the task; despite much effort, the oceanography of the surf zone remains highly empirical and poorly understood. This subject is further complicated by the fact that both waves and currents affect the shape of the ocean bottom through the formation of sand bars and other features. The bars in turn affect wave breaking and the dynamics of currents. A complete understanding of the surf zone must include sediment transport, tides, and local winds. This is a good place to end our course!